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PROGRESSIVE LESSONS

IN

APPLIED SCIENCE.

PART II.—SOLIDITY, WEIGHT, AND PRESSURE.

BY

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INTRODUCTION.

In the former Part we were mostly busied with measurements made on paper and having to do only with Surface; and, so to speak only with one flat surface. We now proceed to consider the measurement of real bodies or *solids* as they are called.

It is not possible for us now to mark the things of which we are treating on paper; all that we can put thereon are *pictures* of the objects, which pictures may serve to bring back to mind the realities, if these realities have been previously seen or understood. Wherefore the Student must endeavour to make up models, and will thus need *tools* and material; for the most part these tools are simple, and the materials easily obtained.

Since solids bounded by flat surfaces come first to be considered, we may procure lumps of some softish material, as *chalk*, *stucco* (plaster of Paris), soft wood, modelling clay, which may be easily pared with a common knife. The chalk or the stucco may be flattened by being rubbed on a piece of flat sandstone, or on a sheet of the "glass paper" used by cabinet-makers, which sheet may be stretched on a flat board. In this way we may get sur-

faces sufficiently flat for our present purpose; the processes for making truly flat surfaces will be afterwards discussed.

When modelling clay is used, it may be smoothed by means of a straight-edged trowel, or be pressed against a flat surface. Those who can use the saw, the chisel, the plane, or the file may prefer to work in harder material.

A flat surface is said to be plane; we often speak of it as a *plane;* but by the word *plane*, as used in geometry, we mean an ideal excessively thin flat sheet (of paper, of glass, of mica as it were), extended in all directions, yet without materiality; so that we may imagine two such planes to cross each other.

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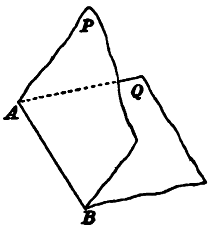
PART II.—SOLIDITY, WEIGHT, AND PRESSURE.

LESSON I.

ON THE MEETING OF TWO PLANE SURFACES.

After having made one face flat upon a lump say of plaster of Paris, if we proceed to cut another flat surface so as to meet the former one, the meeting takes place along a straight line, and forms what we call an *edge*. All our working straight-edges are formed by the meeting of two flat surfaces.



The edge may be blunt or sharp; presenting in this respect characters analogous to those of the angle, and capable of being measured in the same way.

In order to get a clear notion of the matter, let us fancy a straight line A B, say an exceedingly fine steel wire, held in the air, as between the centre-points of a turning-lathe. Let there be attached to this line or wire, a thin flat membrane or *plane* P. The line A B

being kept fixed in position, the membrane or plane may move into the position A B Q, and, continuing its motion, may come back into its old position A B P, having then made a complete turn or *rotation*. Thus edges may be compared with the whole turn, just as angles were; and we have acute, right and obtuse edges just as we had acute, right and obtuse angles.

The student may form in stucco or in soft wood several edges by the meeting of two flat surfaces, making some of them obtuse, some acute.

We form an edge while setting a knife or chisel; this very simple operation is often stupidly performed. Many work away at the very edge, from laziness, not observing that thereby they make the edge blunter every time. In order to keep the tool to the proper degree of sharpness we must grind away as much at the back as at the front; at the thick back of a razor as at the thin edge. In lifting the cutter from the set-stone many workmen thoughtlessly draw the tool towards them so as to make the edge scrape the stone. In sharpening a razor or a penknife, we should turn it over on the back.

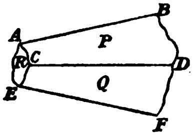
LESSON II.

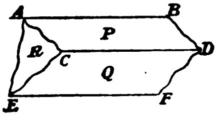
ON THE MEETINGS OF THREE PLANES.

If, after having formed two flat surfaces on a lump of chalk, meeting along some line which we shall call A B, we form a third flat surface meeting both of the former, the one along C D and the other along E F, these three intersections A B, C D, E F, either tend all to one point or are all parallel to each other.

The two lines B A, D C, being both in one plane,

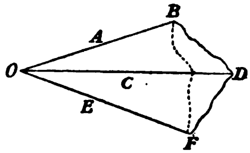
which we shall name P, must meet each other or be parallel; if, when continued, they meet in some point O, that point must be in the extensions of each of the planes Q and R and therefore must be in the prolongation of the third line F E, so that if two of the three common sections of three planes meet each other, the third common section must meet them at the same point.





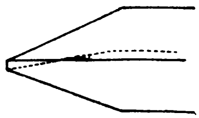
Hence if A B be parallel to C D, E F must be parallel to each of these, for if E F could meet C D, A B would also meet C D at the same point.

If the three planes be extended to meet at O, or if the material be cut away until the three flat surfaces meet each other in a point, there is formed what we shall call a *corner*, it was called by the ancient geometers a *solid angle.* The word *corner* is more expressive and more convenient. A corner, then, is formed by the meeting of three (or more) planes in one point.



The corner at O is bounded by three angles B O D, D O F, F O B, and has connected with it three edges, viz. those on A B, on C D and on E F respectively.

When we wish to make a finely-pointed sharp tool, such as a graver for cutting delicate lines on copperplate, we grind *three* faces to meet each other, not *four;* a four-faced graver is sure to have a double point. If after having ground three faces to meet, we proceed to grind the fourth, we may fall short of or go beyond the mark, in either of which cases there are formed two corners



connected by a short line. Nothing short of absolute perfection in workmanship can give us a point with four faces.

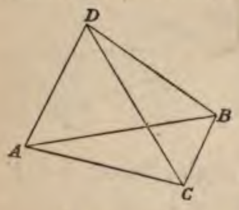
A three-faced point on the end of a hardened steel wire is a useful tool for making centre marks in metal or for piercing minute holes in thin plates.

The student may form corners of various kinds in lumps of clay or stucco. He may also try to form a four or a five faced corner.

LESSON III.

ON THE FOUR-FACED SOLID OR TETRAHEDRON.

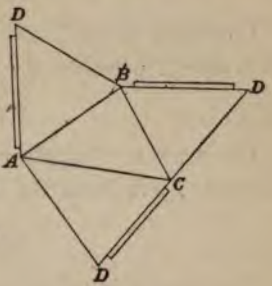
If after having formed three flat faces to meet each other, we cut a fourth flat across all of these three, we shall get a solid bounded entirely by flat faces. No solid can be completely bounded by fewer than four plane surfaces; and



thus the simplest of all flat-faced solids is the *tetrahedron* (four seats) which has four corners, six edges, twelve angles, and six sides. It is a solid occurring very seldom in mechanical operations, and being unfamiliar needs to be the more carefully studied. Actual construction by the student himself is of more use to him than any amount of written description can be. A very convenient construction is to build up the tetrahedron in cardboard, which is cheap and easily procured.

When the lengths of the six sides A B, B C, C A, A D, B D, C D are given, the surface of the tetrahedron may be *marked* off on paper thus:—Let us suppose the actual *solid to be placed* with its face A B C upon the paper; its

apex D being of course above; and let now each of the triangles A D B, B D C, C D A be folded out, as it were, on the cardboard: the figure shown in the margin will result. If now the three lines A B, B C, C A be cut half through the card, while the outline A D B D C D A is cut completely through; the three outer trigons may be folded backwards until the three corners marked D come together and so form the tetrahedron.



In order to keep these together it is proper to leave three projecting flanges, as shown in the drawing; one half of the thickness of the card should be removed from each flange, and the corresponding half thickness should be pared from the inside of the face that is to meet it. A little strong gum or clear glue applied to the flanges before the model is shut up, effectually secures it.

A model prepared in this way wants neatness along the edges. Any joiner or cabinet-maker may, by its help, construct a model in veneer, taking his bevels from the paper model. A tinsmith will have no difficulty in putting the thing together, using his own material.

From this example we see that, in order to determine the very simplest solid, six measurements at the least must be made. They need not all be lines, some of them may be angles, but in all cases there must be data sufficient to lead to a knowledge of the lengths of the six lines.

When the six sides of a tetrahedron are all of one length, each of its four faces is an equilateral trigon, and the solid *is said to be regular*. In this case the edges, angles, and

corners are all alike. The student may count in how many different ways the regular tetrahedron may be replaced.

Exercise 1.

Construct in cardboard two regular tetrahedrons having each side 2 inches long.

Exercise 2.

Construct a tetrahedron having $AB=2\cdot1$; $BC=1\cdot7$; $CA=2\cdot7$; $AD=2\cdot3$; $BD=2\cdot2$; $CD=1\cdot5$ inch.

Exercise 3.

Construct a tetrahedron to the dimensions $AB=2\cdot40$, $AC=1\cdot17$, $AD=\cdot44$, $CD=1\cdot25$, $DB=2\cdot44$, $BC=2\cdot67$ inches. Here each of the angles at A is right, the edges on A B, A C, A D are right edges, and the corner A is what is called a right corner.

Exercise 4.

Construct a tetrahedron having $AB=BC=CD=DA=2\cdot0$; $AC=BD=2\cdot5$.

Exercise 5.

Make a tetrahedron to the dimensions $AB=BC=2\cdot0$; $CD=DA=2\cdot4$; $AC=1\cdot7$, $DB=2\cdot6$.

Exercise 6.

Construct the tetrahedron $AB=BC=CA=2\cdot3$; $AD=BD=CD=1\cdot7$.

Exercise 7.

Construct a solid to the measurements, $AB=2\cdot9$, $AC=2\cdot5$, $AD=1\cdot9$; $BAC=73°$; $CAD=62°$; $DAB=51°$.

LESSON IV.

ON TWIN FORMS.

When we have taken an irregular tetrahedron A B C D, as in the preceding lesson, placed it on the cardboard and proceeded according to the instructions there given, we must fold the three side trigons A D B, B D C, C D A, *downwards* to meet *below* the plane of the paper. The model thus formed is *not* a model of the original solid; we shall find, on trial, that we cannot put the cardboard model into the place which had been occupied by the solid; we cannot even set it down on the same base, for on attempting to do so, we must reverse the scalene trigon A B C face for face; in which position it cannot be replaced.

The student must carefully examine this matter; the ancient geometers seem to have been quite unacquainted with this circumstance and, notably, Euclid proceeds in his argument as if there were no such difficulty: he places the one instead of the other without examination, so rendering all his arguments futile.

The French geometer Legendre was the first who has systematically treated of this peculiar relationship of the two solids, to which relationship he gave the name *symmetry*, from the Greek words *συν* and *μετρεω*, meaning *together* and *measurement;* because each face of the one may be applied to the corresponding face of the other. He calls the two solids *symmetric* to each other; I prefer to use the expressive word *twin*, which is already used by mineralogists in the descriptive title *twin-crystals.*

If we construct two scalene tetrahedrons to the same dimensions, placing the trigon A B C of the one in the order from left to right, and the A B C of the other from *right to left*, we shall form two twin models. And it is to *be remarked* that while we can apply these to each other

face by face and thus examine the equality of their various dimensions; we cannot do the same for two solids exact copies the one of the other.

This twinhood or symmetry is familiarly exemplified in the relation of the right hand to the left, or of the right to the left foot. It is of very frequent occurrence in workmanship.

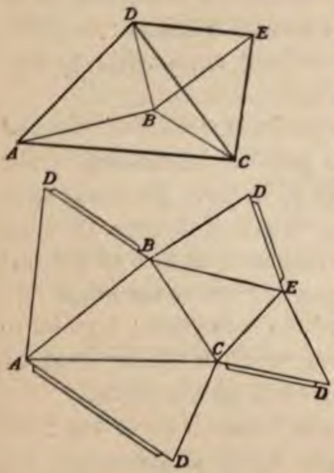
Exercise.

When possible, construct twins to the solids described in the exercises to the preceding lesson.

LESSON V.

ON SOLIDS HAVING FIVE CORNERS.

If we take a point E outside of one of the faces as B C D of a tetrahedron A B C D, and connect this point with each of the sides B C, C D, D B, we shall form a new tetrahedron BCDE, which added to the former one makes up a solid having five corners and six faces: it has also nine edges.



The lengths of the nine sides are sufficient to determine the solid, and the surface may be spread out on cardboard as is shown in the diagram.

This six-faced solid or *hexahedron* has two of its corners, namely those at A and E, formed by the meeting *of three planes;* while each of the corners at B, C, and D,

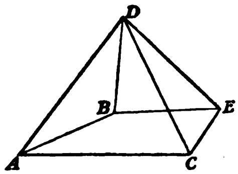
is formed by the meeting of four planes. Hence when we try to make such a form in chalk or stucco, we find it difficult to cut the flat surfaces so as neatly to produce *points* at B, C, D; while there is no such trouble for the corners A and E.

EXERCISE.

Construct twin solids to the dimensions $AB = 2\cdot3$; $AC = 1\cdot9$; $AD = 1\cdot6$; $DC = 2\cdot4$; $CB = 1\cdot5$; $BD = 1\cdot3$; $DE = 1\cdot2$; $BE = 1\cdot1$; $CE = 1\cdot7$.

If we take the fifth point E in the plane of one of the faces, say in the extension of A B C, the two trigons A B C, C B E come to make up one four-sided face, the line C B ceasing to be a side of the solid.

The result is a *pyramid* having the tetragon A B E C for its base and the point D for its apex. This solid having five faces is called a *pentahedron*, of these faces four are trigonal, one tetragonal; it has five corners, of which four are formed by the meeting of three planes each or are *trihedral*, and one is *tetrahedral* or formed by the meeting of four planes.



If the form of the base A B E C be given, and also the three sides A D, B D, C D, the solid is determined. Now to fix the base A B E C we need *five* measurements, and thus we might suppose that eight dimensions are enough to determine the pyramid: whereas in fact *nine* measurements are absolutely needed to determine any solid having five corners. The careful student may try to find out what *ninth* measurement has here escaped notice.

Exercise 1.

Construct a pyramid on a square base A B E C, having A B = B E = E C = C A = A D = B D = C D = E D = 2·0 inches.

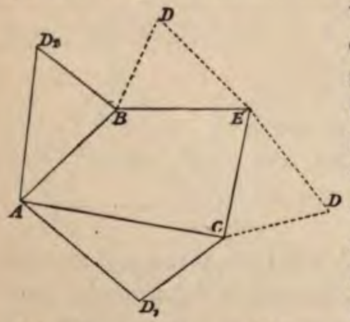
Exercise 2.

Let A B E C be a rectangle having A B = 1·7 ; B E = 2·3, and A D = B D = E D = C D = 2·0.

Exercise 3.

Let A B E C be a rectangle having A B = 1·6 ; B E = 2·5 ; A D = B D = 1·8 ; E D = C D = 1·5.

The development of the surface of a pyramid on cardboard is attended with some difficulty, thus if the tetragon A B E C be given and also the three lines A D, B D, C D,



we are able to prepare in cardboard so much of the surface ; we can construct the trigons AD_2B, AD_1C; but not knowing, as yet, the length of the line E D we are unable to form the remaining faces B D E, E D C. Hence in order to complete our knowledge of the four-sided pyramid we must return to the consideration of edges and corners.

Having cut out so much of the shape, we might fold up the triangles CD_1A, AD_2B to meet, and then measure the distance E D. This is the kind of proceeding to which carpenters are forced to resort when they are unable *to draw out* or to compute the dimensions.

Exercise 4.

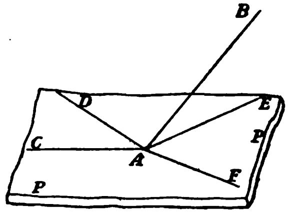
A B E C being a square on 2·0 inches, while B D= 2·3 ; A D=2·1 ; C D=1·9 , obtain E D in this way and construct te pyramid.

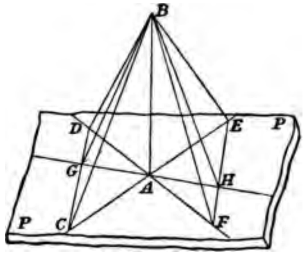
LESSON VI.

ON THE NORMAL TO A PLANE SURFACE.

If we prepare the surface P of a board quite flat, and take any point A in that surface, we may draw from A, in the air, a multitude of lines ; one of these A B may be represented by a needle stuck into the board.

From the point A, let us now draw a number of lines A C, A D, A E, A F all on the board or, as we say, all in the plane P ; and we shall find that the angles B A C, B A D, B A E, B A F are various. It may happen that two of them are alike, but we shall never find three of them to be alike excepting in the particular case which we shall now proceed to study.





Let the line A B be placed so that each of the angles B A C, B A D may be right ; if we produce the lines C A, D A till A E = C A, A F = D A, and join C B, D B, E B, F B, which may be done by fine threads passed through *the eye of the* needle, the tetrahedrons A C D B, A E F D

are exactly alike; the one may be placed instead of the other.

Let now a third line G A H be drawn through the point A and be in the plane P, it is clear that by placing the tetrahedron A E F D in the position of A C D B, the angle B A H would come into the position B A G, so that each of these angles must be right. Thus we arrive at a most important *theorem* which is that:—

If a line be perpendicular to each of two lines in a plane surface, which meet it, it is perpendicular to every line in that surface.

To indicate this perpendicularity of a line to a surface we use the word *normal*, from *norma* the Greek name of the mason's plummet; it has in truth the very same meaning with the Latin word *perpendicular*, but it is convenient to use the one for the relation between two lines or between two planes, the other for the relation of a line to a plane.

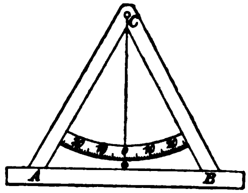
PORISM.

This theorem contains the principle of the common process for producing a flat surface on the turning-lathe. It is quite clear that if the right angle B A C were turned round on A B as an axis, the side A C would sweep over the plane surface P. Hence if the cross slide of a lathe be set accurately square to the spindle, the cutter carried by the slide will make a flat surface on the object fixed to the spindle.

It also contains the principle of the ordinary method of levelling a flat stone.

A very useful form of the mason's plummet-level is shown in the adjoining figure. A B is a straight-edge from which there spring two oblique struts A C, B C *firmly joined* at C. A circular arc having C for its centre

is secured between the two struts and is graduated; from the point C a cord depends carrying a piece of lead which is free to swing below the lower edge of the arc. When the under surface of A B is horizontal the cord should be at the zero of the scale; and the accuracy of the zero may be tested by reversing the instrument face for face. By help of the graduations we are able to lay a stone at any desired inclination, as when adjusting the slope of a pavement so as to throw off the water.

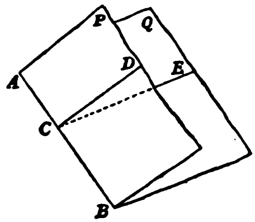


In whatever way a stone may be placed, we can always find one direction upon it which is horizontal; hence the need for trying the level in two directions crossing each other, in order to make sure that the surface is really level.

LESSON VII.

ON THE MEASUREMENT OF EDGES.

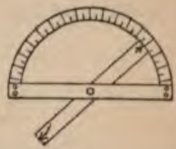
Returning to the figure in Lesson I., if we take any point C in the line A B, and draw, in the plane P, a line C D perpendicular to A B, that line will describe a plane surface when the plane P is made to rotate on A B as an axis; so that if, in the plane Q, C E be made also perpendicular to A B, C E will be one of the positions occupied by C D while turning. Hence while the moving plane passes from the position P into the position Q, the *moving line describes* the angle D C E.



From this it is clear that the edge P, A B, Q formed by the two planes, bears to a whole rotation the same ratio which the angle D C E bears to a whole turn.

The angle D C E is said to be *homologous* to the edge (of the same number with it); and we measure edges in degrees just as we measure angles.

When we wish to measure the edge formed on a solid body by two flat faces, we take a point in the line of intersection or axis of the edge and thence raise two perpendiculars, one in each face: the angle made by these perpendiculars is of the same number of degrees with the edge. In order to measure this angle we use an instrument called the *bevel-gauge*, which is simply a jointed rule somewhat stiff at the joint. This is applied to the two

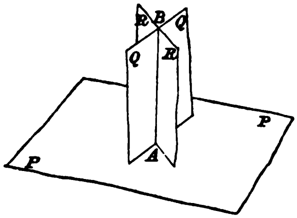


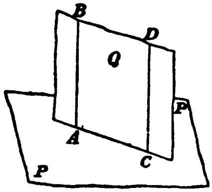
surfaces and is opened out so as to fit on them. The angle of the two rules is then transferred to paper or to a board, and measured with the protractor. Sometimes a graduated arc is attached to the bevel-gauge and the instrument is then called a *goniometer* or edge-measurer. In using these instruments we must be careful to place them square to the edge.

The student may find excellent exercise in measuring the edges of the various solids constructed in accordance with the preceding exercises.

The two faces of a right edge are said to be *perpendicular* to each other. If the line A B be set up normal *to the plane* P, and if a second plane Q be laid along A B,

these two planes are perpendicular to each other; and if two planes Q and R, each being perpendicular to the plane P, meet each other their common section is normal to that plane.



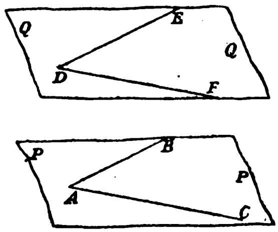


Again, if two lines A B and C D be each normal to the plane P, a second plane Q laid along A B and through the point C will also lie along C D, and will form with the plane P, two right edges, one on each face.

LESSON VIII.

ON PARALLEL PLANES.

If, in the plane P, we draw two inclined lines A B, A C, and, having taken a point D not in the plane P, if we make D E parallel to A B, D F parallel to A C; then a plane Q laid along the two lines DE, DF is parallel to P, that is cannot meet P, however far the two planes may be extended.



We must here be careful to form a clear notion of what is meant by saying that two lines are parallel to each *other. It does* not mean only that the two lines do

meet, for in the figure of Lesson III., the two lines A B and C D do not meet, although they appear, in the drawing, to cross. In order to be parallel, two lines must be in one plane; that is to say in our present figure, a plane passing along the line A B and through the point D, must also pass along D E; and if D E and A B, thus being in one plane, do not meet they are parallel.

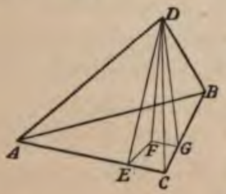
If now the two planes P and Q could meet, they would meet along a straight line, which line, being in the plane P, would meet one or both of the lines A B, A C; if it were to meet A B in some point O, D E and A B would also meet in O, since O is common to the three planes. That is to say, D E could not have been parallel to A B.

Hence the intersections of two parallel planes by a third plane are parallel; hence also if a straight line be normal to each of two planes, those planes are parallel.

LESSON IX.

TO DRAW A NORMAL TO ONE FACE OF A TETRAHEDRON FROM THE OPPOSITE CORNER.

Let it be required to draw from the corner D, a normal to the opposite face A B C of the tetrahedron A B C D.



In the face A D C draw D E perpendicular to A C, and at E in the plane A B C draw E F also perpendicular to A C, then the plane passing along E D and E F is perpendicular to both faces. In the same way draw D G perpendicular to B C, and at G make G F also perpendicular to B C, continuing it to meet E F in F. The plane of

D G F is also perpendicular to A B C, and therefore D F, the intersection of these two planes, is normal to A B C.

In this way we are enabled to mark the point F on the cardboard developme... the surface of the tetrahedron.

The distances D E, E F being thus found, we may construct a right-angled trigon to the dimensions of D E F and so obtain the angle D E F homologous to the edge on A C, which edge we may agree to write D, A C, B; that is the edge formed on the axis A C by planes passing through D and B. In the same way we get the angle D G F homologous to the edge D, B C, A.

If we were to draw D H perpendicular to A B, and to join F H, F H also would be perpendicular to A B, and the angle D H F would serve to measure the edge D, A B, C.

The same construction also enables us to discover the length of the normal D F, which is the *altitude* of the tetrahedron for the base A B C. And in the same way we may get the edges on A D, on B D and on C D as well as the normals drawn from A, B and C to the opposite faces.

Exercises.

Mark, on the tetrahedrons already made, the point at which the normal from each corner falls on the opposite face.

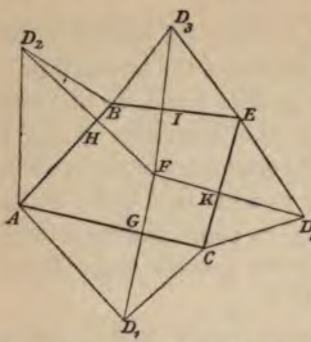
LESSON X.

TO MARK OUT THE SURFACE OF A PYRAMID.

We are now able to finish the construction of the four-sided pyramid begun in Lesson V.

If *from* D_1 we draw $D_1 G$ perpendicular to A C, from

D_2, D_2H perpendicular to AB, and continue these lines to meet, we obtain F, the foot of the normal let fall from the summit of the pyramid upon its base. Draw now FI perpendicular to BE, FK perpendicular to EC and prolong them. From B with the radius BD_2 sweep an arc cutting the continuation of FI at D_3, from C sweep an arc with the radius CD_1, cutting the extension of FK in D_4, then BD_3E, ED_4C are the remaining faces of the pyramid.



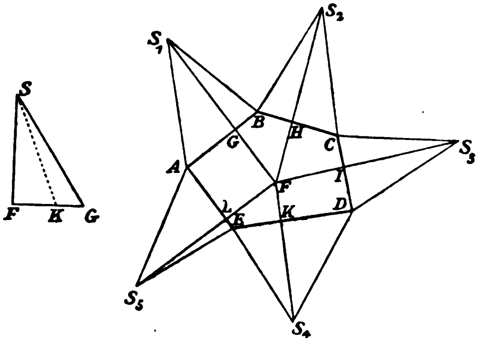
EXERCISE 1.

The base $ABEC$ being determined by the measurements $AB=144$, $BE=221$, $EC=130$, $CA=255$, $AE=275$, and the three slopes CD, BD, AD being each 206, construct the pyramid and its twin.

Hence if the base of a pyramid be given, if the point at which the normal is to fall and the length of that normal be known, we can construct the surfaces of the solid.

Thus, if we were required to construct a pyramid on the pentagonal base $ABCDE$, and having its summit at a prescribed height exactly above the point F, we should draw from F the perpendiculars FG, FH, FI, FK, FL and continue each of them beyond. Then having made a right angle aside at F, we should lay off FS equal to the prescribed altitude of the pyramid, and FG equal to the line FG of our principal figure, then joining GS we obtain the actual distance from the point G to the

summit of the pyramid. Let now $G\,S_1$ be laid off equal to this $G\,S$ and we get the face $A\,S_1\,B$. Inflect now $B\,S_2 = B\,S_1$; $C\,S_3 = C\,S_2$; $D\,S_4 = D\,S_3$ and $E\,S_5 = E\,S_4$, and the remaining four faces are defined.



Here, clearly, we have to satisfy ourselves that the distance $A\,S_5$ is equal to $A\,S_1$ since otherwise the faces would not fold up to meet in one point S.

Now we have seen (Lesson XLIX., Part I.) that the difference between the squares of the two sides of a trigon is equivalent to the difference between the squares of the segments of the third side made by the perpendicular, or that $A\,S_1^2 - F\,A^2 = G\,S_1^2 - F\,G^2$ in the trigon $F\,A\,S_1$; but again in the trigon $F\,B\,S_1$, $G\,S_1^2 - F\,G^2 = B\,S_1^2 - F\,B^2 = B\,S_2^2 - F\,B^2$, and thus the same difference of squares continues all the way round, so that, ultimately, $A\,S_5^2 - F\,A^2 = A\,S_1^2 - F\,A^2$ and consequently $A\,S_5 = A\,S_1$.

The angle F G S in the side figure gives the edge on A B; to find the bevel on any of the other edges, say on D E, we have only to make F K=F K, and to join K S, the angle F K S is homologous to the edge S, D E, B.

Exercise 2.

On a regular pentagon construct a pyramid having for each of its sloped faces an equilateral trigon.

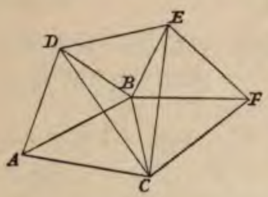
Exercise 3.

On a regular pentagon construct a pyramid having its apex right over the intersection of two of the diagonals of the base, and having its altitude equal to the side of the base.

LESSON XI.

ON SIX-CORNERED SOLIDS.

If we prepare a five-cornered solid bounded by six trigons, as in the first figure of Lesson V., and assume a point outside of one of the faces, to become the apex of a tetrahedron built upon that face, we shall form a solid having



six corners and eight triangular faces; because while three new trigons have been added, one trigon has been covered up; the total number of trigons has been augmented by two. The total number of angles is twenty-four, and these are distributed among the corners thus: two corners, A and F, are trihedral; two, D and E, are tetrahedral; and two, B and C, are pentahedral or formed by the meeting of five angles. Such forms very seldom occur in business.

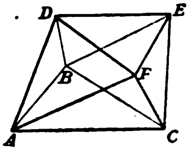
Exercise 1.

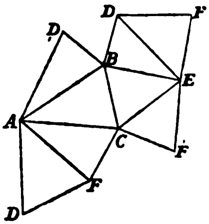
If the student construct a solid of this kind with all its lines of one length he will find the edge on B C to be re-entrant.

Exercise 2.

If all the lines, excepting D E, be alike and if D E be taken four-sevenths of one of the others, the trigons A B C, C F B are in almost one plane.

Instead of taking the new corner outside of one of the faces, we may take it outside of one of the edges as D C, and connect it with the four outlying lines A D, D E, E C, C A thus obtaining four new trigons ADF, DFE, EFC, CFA but covering up two old ones, namely A D C, C D E. In this case also the number of the trigons is augmented by two; but now all the corners are tetrahedral. Each of the eight trigons has three sides, making in all twenty-four sides; but then each edge is formed by the meeting of two faces, wherefore the number of edges is the half of twenty-four, that is twelve. The lengths of these twelve sides are sufficient to determine the form: and when all the twelve are of one length the solid is *regular*, being bounded by eight equilateral trigons uniformly arranged; this is the *regular octahedron*, a form which occurs among crystallized bodies as we shall afterwards see; the mode of spreading out the surface of an octahedron in cardboard is shown in the accompanying drawing.





EXERCISE 3.

Construct a regular octahedron on a side of 2 inches; this is companion to the two regular tetrahedrons previously made; these tetrahedrons placed on opposite faces of the octahedron make up a solid bounded by six rhombuses.

EXERCISE 4.

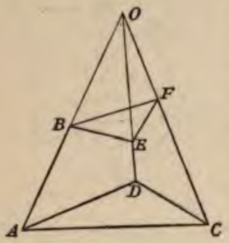
Assume at will twelve unequal lines for the sides of an octahedron; construct it and its twin.

Exercise 5.

If the lines A F, F D, D E, E B, B C, C A be taken all of one length; and the remaining lines A B, B D, D A, C F, FE, EC, all of another length, a semi-regular solid is formed.

In preparing the cardboard for being folded up, a flange should be left on each alternate free edge.

Six-cornered solids may be modified in various ways; thus, returning to the former figure, the two faces A B C, B C F may be extended in one plane so as to form a tetragon, in which case the solid would have seven faces. Or two of such extensions may occur at once, or even three of them, thus giving rise to a great variety of arrangements. Of these only two are of general utility.



If we suppose the edges A, B C, F; D, E C, F and A, D B, E each to become 180°, the solid is bounded by three tetragons A B F C, F C D E, D E B A along with two trigons A D C and B E F, and becomes *pentahedral* or five faced. In this case the three lines A B, D E, C F, being the intersections of three planes, must either tend all to one point or be parallel to each other. If these lines all tend to one point O the solid may be regarded as the remainder obtained by first making a tetrahedron A D C O, and then cutting from it the smaller tetrahedron B E F O.

Exercise.

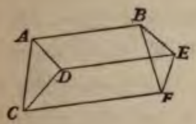
Construct a truncated tetrahedron having B, E, F in the middles of A O, D O, C O respectively. In this case the plane B E F is parallel to A D C.

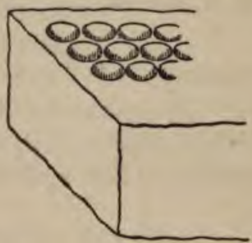
LESSON XII.

ON THE PRISM.

But if the three lines A B, D E, C F be all parallel to each other, the solid becomes what is called a *prism*. Commonly we call the three tetragons the *sides* of the prism, the two trigons its *ends*.

The name is from the Greek word πρὶσμα, saw-scrap; and the application of the term will be understood

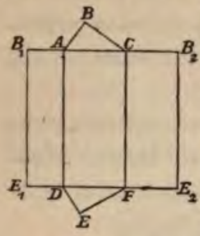




by considering the process of digging out the hollow stone chests or coffins, *sarcophagi*, used by the ancient Egyptians for interment. The operation was performed by means of a thin copper tube which was twirled backwards and forwards against the stone, the end of the tube being supplied with emery and water. In this way a deep thin cylindric cut was made having a solid cylinder of stone in its middle. Rows of such cuts were made close together, leaving between each three contiguous cuts, a three-cornered piece of scrap; the scrap (prisma) and the long cylinders were then broken over; hence the word *prism* comes to be applied by geometers to a th sided piece of scrap, or to a form resembling it.

It is convenient, for our purpose, to confine the the name *prism* to those having their ends parallel,

apply the name *prismoid* (like a prism) to other cases. When the ends of the prism are perpendicular to the sides it is called a *right prism;* and a cut made across any prism or prismoid perpendicularly to the sides is called its *cross-section*, so that the end of a right prism is also its cross-section.

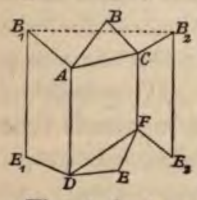


When the length of a right prism and one of the ends are known it is easy to construct it in cardboard. Thus if A B C be one of the ends, we may make the right angle C A D and measure A D equal to the proposed length, then the rectangle A D F C is one of the side faces of the prism. By continuing A C and D F each way till $AB_1 = AB$; $CB_2 = CB$, completing the rectangles, and constructing D E F equal to A B C, we get the remaining faces.

EXERCISE 1.

Construct a right prism of any convenient dimensions.

In the case of an oblique prism or of a prismoid the angles D A C, D A B, &c., are not necessarily right, nor is the line $B_1 A C B_2$ straight, but, in all cases, the lines joining the extreme points B_1, B_2, E_1, E_2 must be perpendicular to A D, for otherwise the points B_1, E_1 will not close with B_2, E_2 when the parts are folded up to form the solid.



The student would do well to examine how it happens that a right prism is equal to its twin, while an oblique scalene prism is not so.

EXERCISE 2.

Construct a prismoid of any convenient form.

LESSON XIII.

ON SEVEN-CORNERED SOLIDS.

PROCEEDING still to add another corner we must augment the number of trigons by two, so that the *seven*-cornered solid must be bounded by *ten* trigons. These may be all in different planes, or some of the adjoining trigons may be in one plane, so that there is a great variety of such solids. Hardly any of them occur in business.

In the case of the seven-cornered decahedron every face is triangular, hence the total number of angles is *thirty;* these cannot possibly be distributed equably among the seven corners, so that there cannot be a regular solid having seven corners. The number of edges is fifteen, so that fifteen distinct measurements or *data* are needed to determine the form.

The ambitious student may exercise himself in constructing seven-cornered solids with various arrangements.

LESSON XIV.

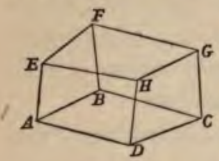
ON EIGHT-CORNERED SOLIDS.

THE addition of another corner gives us two more trigons; the eight-cornered solid then has altogether twelve trigons on its surface; of these some may be in one plane, so that the variety of arrangements is very great.

The twelve trigons may be placed two and two in one plane thus making six tetragonal faces. In this case there are twenty-four angles which may be distributed three and three to each corner, three being the smallest number of

angles which can form a corner. This solid then has all its faces tetragonal, and all its corners trihedral. We can hardly find a name for this form of solid better than the English word *block*.

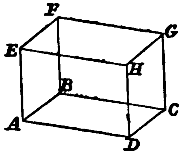
To lay out on cardboard the faces of a flat-faced block is by no means an easy matter when we have to do with irregular forms. If the three tetragons which meet at one corner be given, the solid is determined, thus if A B C D, AEHD, and AEFB be given, we may cut these in cardboard and bring them together, the form of the corner at A being thus quite settled. If the student do this, he will at once see that the position of the plane H D C G is fixed by the two lines D C, D H, while the plane F E H G is settled by E F and E H, wherefore the direction of the intersection H G is determinate; the same being true also of the plane F B C G, it follows that the point G must have a definite position. Hence before constructing the remaining tetragons we must find the means to discover their exact shapes. It is easier to construct such a form in solid wood, or in moulding clay, than in cardboard, and the student having assumed three irregular tetragons round the corner A, may try to discover thence the forms of those meeting at the opposite corner G.



The construction becomes easy when the tetragons are rhomboidal, for then the opposite faces are parallel and equal to each other. A solid of this kind bounded by three pair of parallel faces is called a *parallelopepidon* in the older books, but we shall use the shorter word *rhomb* or rhombohedron.

Every rhomboid which is not a rectangle has two acute and two obtuse angles, hence if none of the faces of a

rhomb be rectangular there must be twelve acute and twelve obtuse angles on its surface. In order to examine how these may be arranged we may begin by supposing the three angles at A to be acute; those at the opposite corner G must also be acute, leaving six acute angles to be distributed among the six corners F, E, H, D, C, B, along with the eight obtuse angles, so that each of these six corners has two obtuse and one acute angle. If again all the angles at A be obtuse, those at G must be obtuse also, and each of the remaining six corners must have two acute and one obtuse angle. Hence oblique rhombs are distinguished into two kinds, one when three acute angles meet, called the *acute rhomb*, the other when three obtuse angles meet, called the *obtuse rhomb*.



Exercise 1.

Construct a rhombohedron having each of the angles at A $101° \cdot 55'$ and all the twelve sides alike.

N.B. This is the form of the fragments of the mineral called *Iceland spar*, and *carbonate of lime*, and will afterwards be of use when we come to study the properties of light; the substance is also called *doubly refracting spar*.

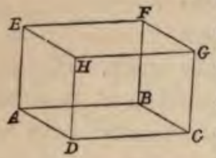
Exercise 2.

Construct also an acute rhombohedron having the angles at A $78° \cdot 05'$ each, and with the same sides as before.

LESSON XV.

ON THE OBLONG AND ON THE CUBE.

When the three angles at A are right, all the other angles are right also and the solid is bounded by three pair of rectangles. This form occurs very frequently in business, more so, perhaps, than any other. We shall give to it the name *oblong;* it has length, A B; breadth, A E; and thickness, A D. It occupies *space*, and, as we shall see anon, is much used in the measurement of space; hence some geometers have described space as having three dimensions.



When the three dimensions of an oblong are all alike, the solid is called a *cube.* We get this word from the Greek κύβος, but it comes originally from the Arabic word كعب (kiub) which takes the plural form كعوب (kiu'oub) in allusion to the fact that all its corners and faces are alike. The cube then is a *regular* solid having eight trihedral corners and six square faces.

A corner formed by the meeting of three right angles is called a *right corner*.

The construction of the cardboard model of an oblong is quite easy.

Exercise 1.

Construct several cubes on 1 inch.

Exercise 2.

Construct an oblong having its sides 4, 5 and 6 parts of any convenient scale. This will be useful in explaining Lesson XVIII.

LESSON XVI.

ON UNIFORMAL SOLIDS.

THE variety in the forms of solids having more than eight corners is so great as to render any discussion fruitless. There are, however, several forms of considerable interest as occurring among minerals or in the course of investigations; two of these deserve our present attention.

When all the corners of a solid are of one kind, and when all the faces are polygons of one name, the solid is said to be *uniformal*. Thus the tetrahedron has all its corners three faced, and all its faces three sided; the octahedron (Lesson XI., Fig. 2) has all its corners four faced and all its faces three sided; and the block (Lesson XIV., Fig. 1) has all its corners three faced and all its faces four sided; these three are examples of uniformal solids.

We have seen that for every additional corner two additional trigons are introduced, wherefore if c denote the number of the corners, $2c-4$ must denote the number of trigons on the whole surface. Putting now f for the number of faces, and supposing that these faces are all of one kind, the number of trigons in one of them must be $\frac{2c-4}{f}$, but the number of corners of a polygon is two more than the number of trigons of which it is composed, wherefore $\frac{2c-4}{f}+2$ is the number of angles on each face; but again the number of faces is f, wherefore the total number of angles on the surface of the solid must be $2c+2f-4$. If all the corners be formed by the meeting of the same number of angles, the above number must be divisible by c, and therefore the two numbers f and c must be such

that $2(c+f) - 4$ is divisible by each of them without remainder; and it is clear that if two such numbers be found, the one of them may be taken for the number of the faces, the other for that of the corners indifferently; moreover, each of these numbers must exceed three. Now the only pairs of numbers satisfying this condition are 4 and 4, 6 and 8, 12 and 20; hence we can only have five kinds of uniformal convex solids, namely the *four-faced* with *four corners*, the *eight-faced* with *six corners*, the *six-faced* with *eight corners*, the *twelve-faced* with *twenty corners*, and the *twenty-faced* with *twelve corners*. We must here carefully limit the statement to *convex solids* because all our attention has hitherto been given to such; we have taken no note of solids having re-entrant edges or corners.

In the first case we have $c = 4$, $f = 4$, whence $2(c+f) - 4 = 12$ is the number of angles, which gives three angles for each corner, three angles for each face; this is the tetrahedron.

In the second case we have $c = 6$, $f = 8$, the total number of angles being twenty-four, giving four angles to each corner, and three angles to each face; the solid being the octahedron figured in Lesson XI., Fig. 2.

In the third case $c = 8$, $f = 6$, the total number of angles being twenty-four as above, giving three angles to each corner, four angles to each face, and producing the hexahedron or block.

In the fourth case $c = 12$, $f = 20$, and $2(c+f) - 4 = 60$ is the number of angles, distributed in fives at each corner, or in threes on each face. This solid then is bounded by twenty trigons meeting five and five at twelve corners. The number of its *sides* or lines is thirty, hence thirty distinct measurements or data are needed for the

determination of the form. The expansion of the surface of such a twenty-faced solid or *icosahedron* on paper is quite easy even when the dimensions are irregular.

In the fifth case $c=20$, $f=12$, the number of angles being sixty as before; of these three go to each corner and five to each face, wherefore the solid is bounded by twelve pentagons meeting three and three at twenty corners. It is a *dodecahedron*. The spreading of the surface of this solid on cardboard for being built up into a model is exceedingly difficult when the dimensions are irregular; the construction of it in stucco or in modelling clay is easy. For the complete determination of the form of a twenty-cornered solid, we require three times twenty less six, that is no less than fifty-four data. Now there are only thirty lines on the solid, wherefore in addition to the lengths of these thirty lines we must have prescribed twenty-four additional data in diagonals, angles or edges.

The student may construct irregular uniformal solids having four, eight and twenty faces in cardboard; and those having six and twelve faces in stucco or clay.

LESSON XVII.

ON REGULAR SOLIDS.

Since no solid can have all its edges re-entrant, and since all the edges of a regular solid are alike, the whole of them must be salient; hence all regular solids are convex, and there cannot exist other than five kinds, namely: the *regular tetrahedron* bounded by four equilateral trigons meeting three and three at four corners; the *regular octahedron* bounded by eight equilateral trigons meeting four and four at six corners; the *regular hexahedron* or

cube bounded by six squares meeting three and three at eight corners; the *regular icosahedron* bounded by twenty equilateral trigons meeting five and five at twelve corners; and the *regular dodecahedron* bounded by twelve regular pentagons meeting three and three at twenty corners.

There is no difficulty in spreading the surfaces of the icosahedron and of the dodecahedron on cardboard, and it is better for the student that he should make out the arrangement for himself than that he should be helped to it by a sketch; he would do well to make a set of the five regular solids all to one length of side.

The cube is the only one of the regular solids which can be used for building up or filling space. The edge of the cube is right, and four cubes may be placed round a line, and therefore eight of them around a point. But the edge of no other regular solid is an aliquot part of a rotation. Although we cannot pack any of the other solids together without leaving interstices, we may fill up space completely by using tetrahedrons and octahedrons combined; for, if we place a tetrahedron on each of two opposite faces of an octahedron we form an acute rhomb, and these rhombs may be packed together.

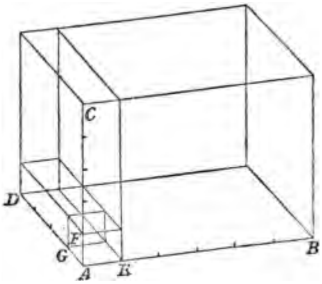
Hence, of absolute necessity, the cube of the linear unit becomes the unit of bulk, and, as has been already stated, the square becomes the unit of surface.

It is to be noticed that these solids go, as it were, in pairs. The middle points, for example, of the faces of the dodecahedron are the corners of an icosahedron and contrary-wise. The middles of the faces of the cube are the corners of an octahedron, the middles of the faces of the octahedron are the corners of a cube. But the middles of the faces of a tetrahedron are the corners of a tetrahedron.

LESSON XVIII.

ON THE MEASUREMENT OF OBLONGS.

WHEN the length, breadth and thickness of an oblong are given in linear units, the number of cubic units in its bulk may readily be computed. Thus if an oblong have its length AB 6 inches, its breadth AC 5 inches and its thickness AD 4 inches, we may measure on A B the distance A E of 1 inch, and through E lead a plane parallel to the plane C A D; thus marking off a slice 1 inch thick, 5 inches long and 4 inches broad. This slice is contained *six* times in the oblong. Measuring along A C the distance A F also 1 inch we lead through F a plane parallel to D A E thus cutting from the above slice a rod 1 inch square by 4 inches in length, which rod is contained *five* times in the slice. Lastly making A G also 1 inch and tracing a plane through G parallel to F A E we mark off a cubic inch, which is contained *four* times in the rod; so that, altogether this cube is contained *four* times *five* times *six* times, that is one hundred and twenty times, in the oblong.



The student is recommended to mark these sections on the paper model previously prepared.

Thus we see that the continued product of the numbers representing the three dimensions, represent the bulk as measured in cubes of the linear unit.

Exercise 1.

A room is 23 feet long, 17 feet broad and 9 feet high; how many cubic feet of air does it contain?

Exercise 2.

A box is 4 feet 7 inches long, by 3 feet 2 inches broad and 2 feet 5 inches deep, how many cubic inches are in its bulk; also how many cubic feet?

Exercise 3.

A box is 4·6 feet long, 3·2 feet broad and 2·4 feet deep, how many cubic feet does it contain?

Exercise 4.

The side of a cube is the third part of an inch, what is the bulk thereof?

Exercise 5.

An oblong piece of iron measures in inches and decimal parts 14·36 by 8·72 by 6·39; required its bulk in cubic inches.

Exercise 6.

An oblong measured in inches and thirty seconds is $14\frac{13}{32}$ by $8\frac{23}{32}$ by $6\frac{15}{32}$; required its bulk in cubic inches.

Exercise 7.

A cube on 75 inches is how many times larger than a cube on 52 inches?

When the three dimensions are alike, the continued product becomes the third power of the number representing one of them, thus a cube on 7 inches contains $7 \times 7 \times 7$ or 343 cubic inches, and a cubic foot contains 12^3 or 1728 cubic inches.

If the dimensions be expressed in parts of the unit, the

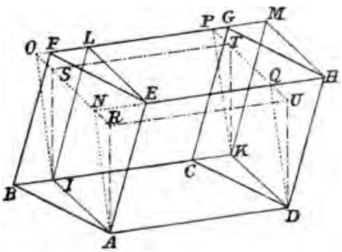
procedure is exactly the same as in the case of rectangles; it is to be observed that the cube on half an inch is the eighth part of the cube on the whole inch, just as the cube of 2 inches is 8 cubic inches. The cube on the tenth part of an inch is the thousandth part of the cubic inch.

LESSON XIX.

ON THE MEASUREMENT OF RHOMBS.

Since we cannot fill up the space of an oblique rhomb with cubes whose edges are right, we must endeavour, by cutting from the rhomb at one place and adding to it as much at another place, to convert it into a right-edged solid. This is just what was done in the case of rhomboids, only here, instead of having one oblique angle, we have three oblique angles to deal with, and so must proceed by steps.

Thus if the rhomb A B C D E F G H have all the angles at A oblique, we may draw in the plane A B C D, A I perpendicular to A D; making D K, E L, H M parallel to it we form the equal prisms ABIEFL, DCKHGM; and, exchanging the one for the other of these, we find the original rhomb to be equivalent to



AIKDELMH which has one of the angles at A right. If now in the plane of A E H D we make A N perpendicular to A D, and complete the rhomb AIKDNOPQ, we have again two equal prisms A N E I O L and D Q H K P M the exchange of which produces the last-named rhomb

which has two of the angles at A right. Lastly, if in the plane of A I O N we make A R, I S perpendicular to A I and draw the corresponding lines D U, K T we obtain a third pair of equal prisms, the exchange of which produces the oblong A I K D R S T U equivalent in volume to the original rhomb.

For convenience of language we may call A B C D the *base* of the rhomb, and A R (the normal to that base included between the two parallel planes) the *altitude* of the solid, and may put the theorem thus: "A rhomb is equivalent to an oblong having the same altitude and standing on an equivalent base;" or we put it in this way: "the solidity of a rhomb is expressed by the product of the numbers representing its base and its altitude;" and, since any one of its faces may be taken as the base, we have three distinct ways of computing the solidity.

Exercise.

Construct a rhomb to the dimensions A B = 1·3; A D = 1·9; A E = 1·2; B A D = 108°; B A E = 95°; E A D = 72°; and measure its solidity in each of the three ways.

N.B. By placing the rhomb on a flat table we may measure the altitude of the overhanging part.

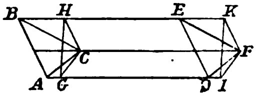
LESSON XX.

ON THE MEASUREMENT OF PRISMS.

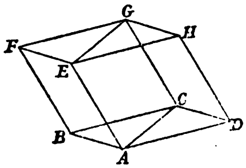
The bulk of an oblique prism is equivalent to that of a right prism having the same length and the same cross-section.

Not more than one of the side faces of an oblique prism can be rectangular, and when not one of them is so, two

oblique and two acute angles occur at the ends of one of the three parallel sides; thus at C there may occur two obtuse and at F two acute angles. Let us now make C G in the plane A C F D, and C H in the plane B C F E each perpendicular to C F, and join H G, so that G C H may be the cross-section of the prism, and in this way cutting off a pyramid having A B H G for its base with C for its apex. On transferring this pyramid to the position D E K I F we convert the oblique into a right prism G H C I K F having the same cross-section and the same length.



The plane A E G C passing along the two opposite parallel sides of a rhomb divides it into two prisms twin to each other. These twin prisms have equal cross-sections and the same length, wherefore they are severally equivalent to right prisms having equal cross-sections and equal lengths. But twin right prisms are equal to each other, wherefore the twin prisms A B C E F G and C D A G H E are equivalent to each other, and each of them is half of the rhomb.



Hence we have three modes for computing the solidity of an oblique prism. We may regard one of the rhomboidal faces, say A B F E, in the last figure, as the base, and the normal drawn from some point in G C to that plane as the altitude; the solidity of the prism A B C E F G is then expressed by half the product of the numbers which represent the base and the altitude. Again we may regard the trigon A B C as the base and the normal included between the two parallel end planes as the altitude. The solidity of the prism is then given by the product of the numbers

representing the base and altitude. Thirdly, we may take the product of the numbers representing the cross-section and the length.

For the sake of shortness we often say "the product of the base by the altitude"; this is called an *ellipsis* (or leaving out) in language; the student must be careful not to be misled by the use of this ellipsis into fancying that we can multiply a surface by a line; the real meaning is that we are to multiply the number representing the one thing by the number representing the other thing.

EXERCISE.

The cross-section of an oblique prism has G C = 25, C'H = 34, H G = 39; the length F C is 58, while G A = 8, H B = 14; compute the lengths of the sides C A, A B, B C'; compute also the areas of the faces and the solidity of the prism. Construct also the prism in cardboard. The area A B C being known the altitude of the prism when set on end may be computed.

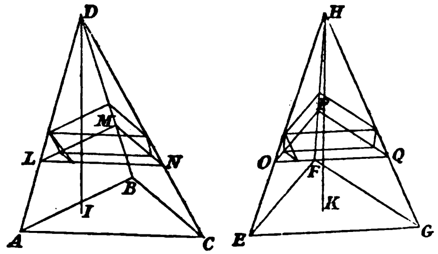
LESSON XXI.

ON THE MEASUREMENT OF TETRAHEDRONS.

ALTHOUGH the tetrahedron be the simplest of all solids in regard to the number of its parts, the computation of its bulk is attended with considerable difficulty; for this reason, that we cannot place equal tetrahedrons together so as to make up a rhomb or a prism.

If we construct two tetrahedrons having equivalent bases A B C and E F G, and having equal altitudes, we infer, by analogy, that their volumes are alike; in order to have a satisfactory demonstration, however, we must have recourse to a somewhat lengthened train of reasoning.

Let us suppose both tetrahedrons placed on the plane of the table and let them both be cut by a series of planes parallel to the table into a number of thin slices, of which it is enough for our purpose to picture one in each solid.



The student would do well to make two tetrahedrons in cardboard and to trace upon their faces all the outer lines of the following construction.

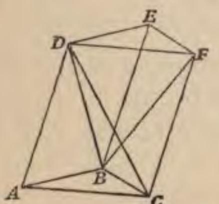
It is obvious that if the plane L M N be parallel to A B C, the trigons L M N, L D N, N D M, M D L must be similar to A B C, A D C, C D B, B D A, and that the tetrahedron L M N D is similar to A B C D ; and in the same way that the solid O P Q H is similar to E F G H. Also since these planes L M N, O P Q are at equal heights from A B C, E F G, the lines A D, B D, C D ; E H, F H, G H must be cut proportionally, and A B : L M :: E F : O P ; wherefore the proportion A B C : L M N :: E F Q : O P Q subsists among the surfaces, in other words the trigon L M N is equivalent to O P Q ; and so of every pair of planes. Now the slice, or space between L M N and the plane above it, may be divided, as shown in the figure, into three parts, one a small tetrahedron similar to A B C D at the corner L ; the second, a prism placed between this tetrahedron and the corner N and having its three parallel sides parallel to A C ; the third a prism placed between this prism and the corner M, and having

its three parallel sides parallel to B D. The corresponding slice of the tetrahedron E F G H is shown as similarly divided.

Now, from what has been shown in regard to the solidity of prisms, it is clear that each of the prisms in the slice of A B C D is equivalent to the corresponding prism in the slice of E F G H, and thus we have *almost* shown that the slices themselves are equivalent; the only want is that we have yet to prove the one of the small tetrahedrons to be equivalent to the other; and this, in strict reasoning, is just as difficult as the original theorem.

However, leaving out of view for the moment these small tetrahedrons, we thus show that the solidities of the various slices of A B C D are equivalent to those of the corresponding slices of E F G H, and hence our demonstration is good for the whole tetrahedrons, excepting the rows of small ones placed along the edges A D and E H.

But each pair of these small solids may be treated in the same way, and thus we may extend our demonstration to the entire solidity less a row of minute notches along one of the sides, which notches may be made indefinitely small; hence we conclude that the tetrahedrons having equal altitudes and standing on equivalent bases are equivalent to each other.



Let us place two lines B E, C F, in the air, parallel and equal to A D, and complete the prism A B C D E F; also let a plane be led through the three points B D F; the prism is then seen to be composed of three tetrahedrons A B C D, D E F B and B C D F, which are equivalent to each other, for the two A B C D, D E F B have equal bases

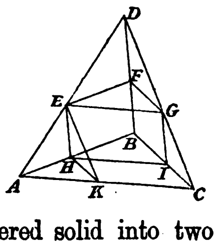
A B C and D E F and have the same altitude since these end planes are parallel; and D E F B and D B C F are on equal bases B C F, F E B, and have the same altitude namely the normal let fall from D upon the plane B E F C. Hence it follows that the solidity of the tetrahedron is the third part of that of the prism standing on the same triangular base and having the same altitude, or, in numbers, the solidity of a tetrahedron is the third part of the product of its base by its altitude.

An expert carpenter may, without much difficulty, construct a dissected model in hard wood: it requires some skill to make the parts join neatly together.

The student would do well to examine very carefully this line of argument; it furnishes instructive examples of the two processes called, in modern scientific works, *differentiation* and *integration,* processes without which it is impossible to study the laws of the motions of bodies.

For the sake of the more ambitious student I shall place the matter in another light.

Having bisected the side A D of the tetrahedron A B C D, in E, draw the plane E F G parallel to A B C, and E H K parallel to D B C, thus cutting off two tetrahedrons E F G D, A H K E each similar to the original solid and of exactly half the dimensions, and leaving a seven-cornered solid H K E B F G C. Lead now a plane along the two lines E G, E H and continue it to cut B C in I, this plane divides the seven-cornered solid into two prisms H B I E F G, and H E K I G C.



The construction of these separate solids in cardboard or in hard wood may furnish a good exercise to the student.

For shortness, let us put b for the area of the base A B C, and a for the altitude or length of the normal drawn from D to the plane A B C.

The solidity of the prism H B I E F G is the product of the base H B I which is one fourth part of A B C, by the altitude which is the half of a, wherefore, in numbers, the solidity is $\frac{1}{4} b \times \frac{1}{2} a = \frac{1}{8} a b$.

Again K H I C is half of the trigon A B C, so that the bulk of the prism K H I C E G, being half the product of its rhomboidal side by the altitude thereon is $\frac{1}{2} \times \frac{1}{2} b \times \frac{1}{2} a = \frac{1}{8} a b$; and thus the sum of the two prisms is $\frac{1}{4} a b$. Hence if we write T for the solidity of the whole tetrahedron, and T_1 for that of A K H E we have

$$T = \tfrac{1}{4} ab + 2 T_1.$$

Proceeding in exactly the same way with one of the smaller tetrahedrons, we observe that its base is one-fourth of A B C, and its altitude one-half of a, wherefore the product of its base by its altitude is one eighth part of b, so that we may write

$$T_1 = \tfrac{1}{32} ab + 2 T_2,$$

putting T_2 for the solidity of the still smaller tetrahedron. From this it follows that

$$T = \tfrac{1}{4} a b + \tfrac{1}{16} a b + 4 T_2.$$

Proceeding another step in the same direction we find

$$T = \tfrac{1}{4} a b + \tfrac{1}{16} a b + \tfrac{1}{64} a b + 8 T_3$$

the smaller tetrahedrons become twice as numerous at each bisection, but much smaller in aggregate bulk; so that if we carry this process to a great length, the row of

notches along the line A D becomes insignificant in volume, and the solidity of the tetrahedron is represented by the interminate series

$$T = ab\left\{\tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \tfrac{1}{256} + \text{etc.}\right\}$$

each term of which is the fourth part of the preceding term; it is one of those progressions which are rather strangely called *geometrical.* On multiplying each side of this equation by 4 we find

$$4\,T = ab\left\{1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \text{etc.}\right\}$$

whence, by subtraction, $3\,T = ab$, $T = \tfrac{1}{3}ab$.

By dividing the line A D into three equal parts and making the corresponding construction, we may obtain the same result by another converging series of fractions.

Since we may regard any face of the tetrahedron as its base, we have four modes of measurement, and it follows that the four altitudes are reciprocally proportional to the areas of the faces on which they fall; that is the longer normal falls on the smaller area.

Exercise 1.

The side of a regular tetrahedron being $1 \cdot 98$, compute the altitude and the solidity thereof.

Exercise 2.

The base being an equilateral trigon on a side of $2 \cdot 00$, and each of the sloped lines being $2 \cdot 40$ inches, compute the altitude and the bulk of the tetrahedron.

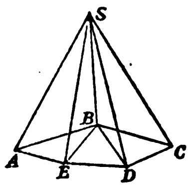
Exercise 3.

The dimensions of a tetrahedron being $AB = BC = CD = DA = 65$; $AC = 78$, $BD = 50$; compute the areas of the faces; the altitudes of the tetrahedron and its solidity.

LESSON XXII.

ON THE MEASUREMENT OF PYRAMIDS.

A PYRAMID having the polygon A B C D E for its base and the point S for its summit may be divided by the planes B S E, B S D, into tetrahedrons having a common altitude, and each of these is the third part of a prism having that altitude, wherefore the solidity of the whole pyramid is the third part of a prism having A B C D E for its base, the same altitude with the pyramid, or, in numbers, the bulk of a pyramid is the third part of the product of its base by its altitude.



EXERCISE 1.

A pyramid is constructed on a square base whose side is 5·77 inches, the sloped faces being equilateral triangles; required the solidity.

N.B. This pyramid is one half of a regular octahedron.

EXERCISE 2.

A pyramid is constructed on a rectangular base whose length is 32 and breadth 24, while the sloped sides are all of the length 25; required the areas of the faces and the solidity of the pyramid.

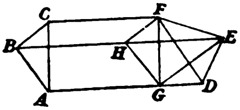
EXERCISE 3.

The base of a pyramid is a rhomboid whose sides are 25 and 33, the longer diagonal being 52, and the altitude of the pyramid (at the middle of the base) being 39; compute the lengths of the slopes, the areas of the faces, and the solidity of the pyramid.

LESSON XXIII.

TO COMPUTE THE SOLIDITY OF A PRISMOID.

LET it be required to compute the solidity of the prismoid A B C D E F, having the three parallels A D, B E, C F of unequal lengths. It will be sufficient for our purpose to suppose that one of the ends, as A B C is perpendicular to A D, or is the cross-section of the solid.



C F being the shortest of the three parallel lines, let a plane F G H be drawn through F parallel to C A B, thus dividing the solid into a right prism A B C G H F and a pyramid having F for its summit and G H E D for its base. This pyramid may be divided into two tetrahedrons E G H F and E D G F by the plane F E G. The former is the third part of a prism having F H G for its cross-section and H E for its height. The latter is equivalent to F H D G which again is equivalent to a prism with the same cross-section and the third part of G D for its height, hence we have

$$\begin{aligned} ABCGHF &= ABC \times CF \\ EGHF &= ABC \times \tfrac{1}{3} HE \\ EDGF &= ABC \times \tfrac{1}{3} GD \end{aligned}$$

and, adding these values together,

$$\begin{aligned} ABCDEF &= ABC \times \{ CF + \tfrac{1}{3} HE + \tfrac{1}{3} GD \} \\ &= ABC \times \tfrac{1}{3} \{ 3CF + HE + GD \} \\ &= ABC \times \tfrac{1}{3} \{ AD + BE + CF \} \end{aligned}$$

that is to say the solidity of a prismoid is equivalent to that of a prism having the same cross-section and having its length equal to the average of the three parallel sides of the prismoid.

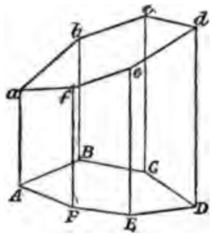
Exercise 1.

The three sides of the cross-section A B C being A B = 86, B C = 75, C A = 97; and the three unequal lengths being A D = 104, B E = 123, C F = 95, compute the lengths of D E, E F, F D; the areas of the faces and the solidity of the prismoid.

Exercise 2.

The sides of the cross-section being A B = 193, B C = 194, C A = 195; while A D = 206, B E = 235, C F = 198; compute E D, D F, F E, the areas of the faces and the solidity of the prismoid.

When the cross-section of a prism is polygonal, its solidity is the product of the area of the entire cross-section by the length; but when the ends are not parallel, as in the figure, we cannot take the average or mean length of the unequal lines A a, B b, C c, D d, E e, F f, as the length of an equivalent right prism, but must divide the columnar solid into prismoids having triangular cross-sections, and treat each of these separately.



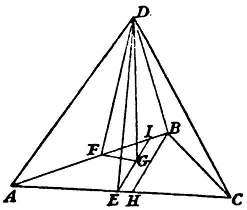
LESSON XXIV.

COMPUTATION OF THE TETRAHEDRON.

All flat-faced solids may be divided into tetrahedrons; hence if we be able to compute the solidity of a tetrahedron, we shall be able to compute that of any polyhedron of given dimensions. The ease or the difficulty of the

computation will depend on the character of the data, that is to say, on the mode of measurement.

The case which naturally comes first is that when the six sides of the tetrahedron are given. Selecting the face A B C as the base we can compute its area by the method explained in Lesson XLIX., Part I.; it remains for us to compute the altitude or the length of the normal D G drawn from the opposite corner.



In order to find the point G we draw the two lines D E, D F perpendicular to A C, A B respectively, and at E and F in the plane of the base raise perpendiculars meeting in the point G. The lengths of A E, E D, A F, F D may all be computed by the methods already explained. Draw B H perpendicular to A C; the distances A H, H B may be computed in the same manner. If now we produce E G to meet A B in I, the trigon A E I thus formed is similar to A H B, wherefore A I and E I may be computed by proportion, and I F got by taking A F from A I. Again, since the angle F I G is equal to H B A and the right angle I F G to B H A, the trigon I F G is similar to B H A, or B H : H A :: I F : F G, wherefore F G may be computed, so that, the hypotenuse D F and the side F G being known, the other side G D of the right angle F G D may be found; and the solidity of the tetrahedron thus obtained.

Exercise.

Compute the areas of the faces and the solidity of a tetrahedron having the dimensions A B = 143, A C = 153, A D = 157, B C = 100, C D = 120, D B = 140.

N.B. The student may have observed that in many cases

the areas of the trigons proposed in the exercises have come out in whole numbers; the arrangement of the numbers so as to produce this result is by no means easy. As yet no method has been discovered for getting numbers such as to make all the areas of the faces of a tetrahedron integer.

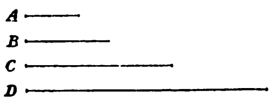
The student may assume various dimensions, and exercise himself in computing the solidities.

LESSON XXV.

ON SIMILAR SOLIDS.

THE proportions among the lines connected with similar solids are quite analogous to those connected with similar figures drawn on a plane. Each line in the one solid is to the corresponding line in the other solid in a fixed ratio, or in the ratio of the scales according to which they are constructed. Thus if a model be made on the scale of one inch to a foot, each line on the model is the twelfth part of the corresponding line on the actual structure. But the surfaces of the model are not merely twelve times, they are twelve times twelve times, that is 144, less than the actual surfaces; while the solidities are twelve times that again, or 1728 times smaller than the reality.

If, to take another example, the scales according to which two solids are made be in the ratio of 3 : 5; each line of the one is to the corresponding line of the other in that ratio; but their surfaces are in the ratio of 9 : 25, and their



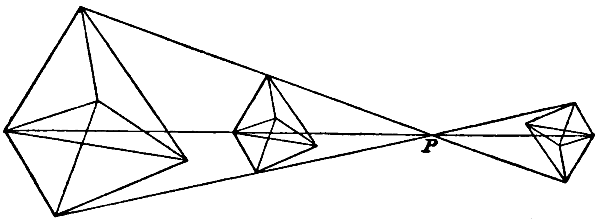
bulks in the still higher ratio of 27 : 125. Thus if the line B be made five-thirds of A, the ratio A : B is that of

the lines of the two solids. If we now make C five-thirds of B, the ratio of A : C is that of the surfaces; and if D be made five-thirds of C, the ratio of the bulks is that of A : D.

This theorem is the most important in the whole science of Geometry.

It shows us that we cannot make two machines or two structures of any kind of different sizes and with the same proportions. As to mere shape, we succeed quite well; but then while we change the weights in one ratio, we change the strengths in another. The strengths are changed in the ratio of A : C, the weights in the ratio of A : D; so that, while keeping to the same drawings, we may make the structure so large as to fall to pieces by its own weight.

If, having taken some point P outside of a polyhedron, we join each corner of the solid with it by a straight line and cut all these lines in the same ratio, the points of section are the corners of a similar solid; but if the lines be



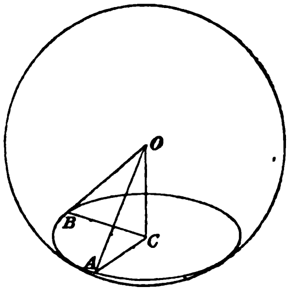
all produced in the same ratio, beyond P, the solid so obtained is similar to the twin of the original polyhedron, as is evident when we consider that the surfaces turned towards us correspond with those turned from us in the first solid.

Hence it is that the images formed in the photographer's camera are right-left of the objects themselves.

LESSON XXVI.

ON THE SPHERE.

If a straight line of a given length have one of its ends at a fixed point while the line is free to turn in every possible direction, the free extremity is in the surface of a sphere. Or if a circle turn upon one of its diameters as an axis, it describes a sphere. Hence a sphere may be turned in its socket in every possible direction; and hence spheres with equal radii are equal to each other. If a plane pass through the centre of a sphere the section is a circle, and the sphere is divided into two halves called *hemispheres*.



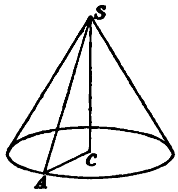
The section of a sphere by any plane whatever is a circle, for if we draw from O, the centre of a sphere, the normal O C to the second plane, and, having taken any two points A and B in the boundary of the section, join A C, B C, we have two right-angled trigons having the hypotenuse O A equal to O B, and O C common, wherefore C A is equal to C B.

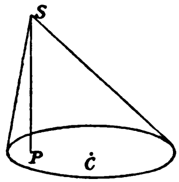
Here we notice that the square of A C, the radius of the section is less than the square of the radius of the sphere by the square of O C. Hence the section made by a plane passing through the centre is greater than any section not passing through it, and is therefore called a *great circle* of the sphere.

LESSON XXVII.

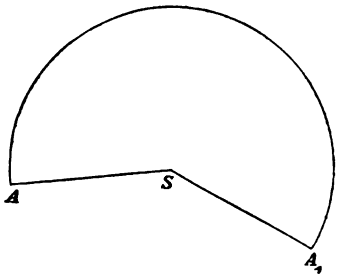
ON THE CONE.

If a point be taken out of the plane of a circle, and if a straight line resting on that point be led round the circumference, the solid thus indicated is called a cone, the fixed point being called the *apex*, and the circle the *base* of the cone. When the line joining the apex of the cone





with the centre of the base is normal thereto, the cone is said to be *right ;* otherwise it is called oblique. We can make a right cone on the turning-lathe, but the construction of an oblique cone is only to be done by hand, and is a pretty severe test of the workman's skill.



If a cone, whether right or oblique, be laid upon a flat board and made to roll, the apex does not change its position, and the conical surface is spread or *developed* on the plane. A piece of paper cut to the form of this development may be wrapped on the cone so as to cover it. The development of a right cone is a circular sector, whose radius *is equal to* the slope S A. If the cone, on being

rolled once round, bring the line S A from the direction S A to S A_1 on the board, the arc must be equal in length to the circumference of the base; wherefore the angle A S A_1 of the development must be to an entire turn as the radius C A of the base of the cone is to the slope S *A*. Hence this angle is easily computed. For example, if the radius of the base be 3 inches, and the height *C S* 4 inches, the slope S A must be 5. Therefore if we describe a circle with 5 inches for its radius, and make the angle A S A_1 three-fifths of 360°, that is 216°, the sector A S A_1 will just cover the conical surface, because an arc of 216° of a circle whose radius is 5 inches is equal in length to the whole circumference of a circle 3 inches in radius.

Exercise 1.

The radius of the base of a right cone being 9 and the altitude 40 , make the development of its curved surface in stiff paper.

Exercise 2.

The radius of the base of a right cone being 24 and the altitude 7, make the development of its curved surface.

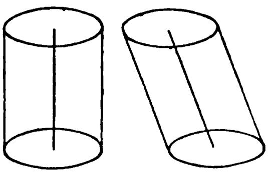
To trace the development of an oblique cone is an exceedingly difficult task; the distances from the apex S are not alike, and therefore the outline of the development is not circular.

If the straight lines A S be extended beyond the apex S, another cone is formed; and the two together constitute the complete conical solid.

LESSON XXVIII.

ON THE CYLINDER.

If a straight line, kept always parallel to a fixed straight line, be carried round the circumference of a circle, the solid thus traced out is called a cylinder (Greek κύλινδρος). The fixed straight line may conveniently be made to pass through the centre of the circle, and is then called the *axis*. When the axis is normal to the plane of the circle, the cylinder is *right*, otherwise it is *oblique*. A right cylinder may be formed on the ordinary turning-lathe; but to form an oblique cylinder we need the instrument called the oval-chuck.



A cylinder may be rolled upon a plane, or *developed;* the development of a right cylinder is a rectangle, having the length of the cylinder for one dimension and the circumference of the base for the other; but the development of an oblique cylinder has waved ends, and is extremely difficult of delineation.

Exercise.

Develop the curved surface of a right cylinder 1·5 inch diameter and 2 inches long.

For the present we shall confine our attention to the right cylinder and the right cone, leaving the oblique ones to be afterwards examined.

LESSON XXIX.

MEASUREMENT OF THE CYLINDER.

If we inscribe a polygon in the base of a cylinder and draw lines parallel to the axis A B through the corners, we obtain a like polygon inscribed in the upper end; and these lines limit a prism inscribed in the cylinder. The solidity of that prism is expressed, in numbers, by the product of the area of the base by the height. Now by making the corners of the polygon more and more numerous we bring its surface to be nearer and nearer to that of the circle, while at the same time the solidity of the prism approaches to that of the cylinder, wherefore we conclude that the solidity of a cylinder is to be got by multiplying together the numbers expressing the area of the base and the altitude.

Hence if R be the radius of the cylinder and H its height, πR^2 is the area of the base (Lesson LVIII., Part I.), and $\pi R^2 H$ is the solidity. This is true also of the oblique cylinder.

Since, in the right cylinder, all the faces of the inscribed prism are rectangular, the total area of the prism surface is the rectangle under the height of the cylinder and the perimeter of the polygon; hence the curved surface of a cylinder is the rectangle under the circumference of the base, and the altitude, and is expressed by the formula $2\pi R H$. But in the oblique cylinder the faces of the *inscribed* prism are rhomboids with varying angles and *this formula* does not hold good.

Exercise 1.

Compute the total surface and the solidity of a cylinder inches in diameter and 5 inches long.

Exercise 2.

Compute the total surface and the solidity of a cylinder 5 inches in diameter and 4 inches long.

Exercise 3.

From a solid cylinder 6 inches long and 3 inches in diameter a hole is bored out 1·8 in diameter, required the solidity of the tube thus made.

Exercise 4.

Required the bulk of the metal in a tube 12 inches long, 2·7 inside diameter, the metal being ·05 thick.

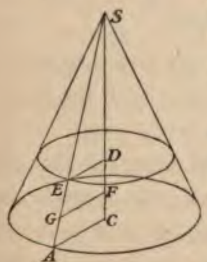
LESSON XXX.

MEASUREMENT OF THE CONE.

If we make a polygon circumscribing the base of a right cone, and join the corners of that polygon with the apex, we form a pyramid whose slopes touch the cone. The solidity of this prism exceeds that of the cone; on making the sides of the polygon more numerous its area approaches to that of the base, while the solidity of the pyramid approaches to that of the cone; hence we conclude that the solidity of a cone is the third part of the product of its base by its altitude; or if R be the radius of the base and H the height, the solidity is given by the formula $\frac{1}{3}\pi R^2 H$; which formula serves also for an oblique cone.

The line S A joining the apex with the point of contact of one of the sides is necessarily perpendicular to that side, now all these perpendiculars are of one length, wherefore the entire sloped surface of the pyramid is equivalent to half the rectangle under S A and the perimeter of the polygon. Wherefore, supposing the number of sides to be indefinitely augmented, the curved surface of the cone is half the rectangle under the slope S A and the circumference of the base, and is therefore given by the formula $\pi R \times SA$. But the square of S A is equivalent to the sum of the squares of S C and of C A, wherefore instead of S A we may write $\surd(H^2 + R^2)$, and thus the expression for the curved surface becomes $\pi R \surd(H^2 + R^2)$. This formula is quite inapplicable to the oblique cone.

When a cone is cut by a plane parallel to its base the section is a circle, and the cone cut off is similar to the original. The surface and the solidity of the truncated portion may be computed by subtraction.



If the cone were rolled on a flat surface the development of the truncated portion would be contained between two concentric circumferences descibed with the radii S A and S E, and is equivalent to a trapezoid having A E for its altitude and the lengths of the two circumferences for its parallel sides; wherefore the area is equivalent to a rectangle under A E and the circum-

ference of a circle described with the radius F G which is half the sum of C A and D E.

EXERCISE 1.

Required the curved and the flat surface of a cone the radius of whose base is 39 and the altitude 80.

EXERCISE 2.

A cone having the radius of its base 48 and its altitude 55, is cut into slices each of the thickness 11, by planes parallel to the base, required the solidity of each of the five parts; as also the curved surface of each.

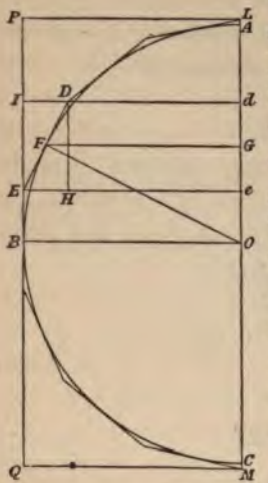
LESSON XXXI.

MEASUREMENT OF THE SPHERE.

WE cannot develop a spherical surface upon a plane; thus if we take a piece of paper and press it to fit on a sphere we find that it is creased at the edges. It is only by using very small pieces, and by wetting them till they become soft, that we can cover a globe with paper.

A semicircle A B C turned upon its diameter A O C as an axis, describes a sphere; let this semicircle be circumscribed by the half of a regular polygon extending from L to M, D E being one of its sides. If this half polygon be made to rotate on the axis L M, its boundary will describe a surface composed of conical pieces; thus the part D E will trace a truncated cone, d D being the radius

of the top, e E being the radius of the base. Bisecting D E at F and drawing F G parallel to E e, the surface of this truncated cone is expressed by 2π.F G.D E. Let us draw D H parallel to A O, and join O F; then the triangle D H E is similar to F G O and E D : D H :: O F : F G wherefore the rectangle F G.D E is equivalent to D H.O F so that the surface described by D E may be written 2π.D H.O F or 2π.O B.I E; now this expresses the cylindric surface described by the line I E, and thus it follows that the conical surface described by D E is equivalent to the cylindric surface described by I E.



This same equivalence holds good all along; wherefore the entire surface described by the half polygon is equivalent to the cylindric surface described by the line P Q which is equal to L M.

But if we increase the number of sides of the polygon, the points L and M will come close to A and C, while the compound conical surface comes close to that of the sphere, and thus it follows that the entire surface of the sphere is equivalent to the curved surface of the cylinder containing it; and also that if the two solids be cut into slices by planes parallel to the ends of the cylinder, the spherical is equivalent to the cylindric surface of each slice.

Since the height of the containing cylinder is 2 R, the curved surface is $4\pi R^2$, wherefore the entire surface of *the sphere is four* times that of its great circle.

If we imagine the surface of a sphere to be divided into a great number of minute parts, and lines to be drawn from the centre to the corners of those parts, the solidity of the sphere will be divided into a multitude of pyramids having those parts for bases and the radius of the sphere for their common altitude; hence the solidity of the sphere is equivalent to that of a pyramid having for its base a surface as extensive as the surface of the sphere and having the radius for its altitude, it is therefore represented by the formula $\frac{4}{3}\pi R^3$; or if D be the diameter, $\frac{1}{6}\pi D^3$. The bulk of the sphere then is rather more than half of the cube from which it may be cut.

EXERCISE 1.

Required the surface and the solidity of a sphere 17 inches in diameter.

EXERCISE 2.

The radius of the earth is very nearly 3960 English miles, what is its surface in square miles and what its solidity in cubic miles?

EXERCISE 3.

How many square inches are in the surface of a globe 21 inches in diameter? and what is the bulk?

EXERCISE 4.

How many cubic inches of metal are needed to make a hollow spherical shell 13 inches in outer diameter and ·47 of an inch thick?

EXERCISE 5.

Required the diameter of a sphere which shall contain exactly one cubic foot.

LESSON XXXII.

ON SLIDING SURFACES.

If a hollow be made in any substance exactly to the shape of a spherical ball, the ball may be set in the hollow and may be moved about in all directions, the two surfaces fitting all the while. This is an example of sliding surfaces. It furnishes the workman with a ready means of forming an exact sphere of any moderately hard material.

For example if we make up a lump of paper pulp with a very little sizing to keep the fibres together, and dig in a piece of pumice-stone a cavity roughly to fit so as to cover say one fourth part of the surface; we may by rubbing the pumice all over the lump, wear away the more prominent parts; at the same time wearing away the prominent parts of the pumice. In this way the two come to fit each other. We may hasten the process by pasting paper on the parts which appear defective; and when the contact is true all over, the ball must be accurately spherical. In conducting this operation it is proper to dig out the centre part of the pumice considerably below the sphere, in order to give a firmer contact. It is essential that both surfaces wear away, for otherwise they could never come to fit.

The same method may be used for hard substances, provided we have some means of wearing or abrading them. This we do by help of a grinding powder such as sand or emery. Having made a ball of glass nearly round by grinding on a lap or grind stone, we prepare a hollow *tool* of brass to fit, removing the centre part, and we pass this over the glass in all directions until every unevenness disappear, keeping the tool well supplied with emery and

water, and using finer emery as the work approaches completion. The whole operation may be performed in the hand without a lathe or any other apparatus. A brass tool answers better than one of lead, because the lead is not worn away so easily, it being essential that the tool itself wear. This operation of grinding is slow; we therefore look for some more expeditious method of making the ball nearly true and have recourse to the grinding only for the last finish.

We have seen that every section of a sphere by a plane is circular; wherefore if the end of a cylindric tube be applied to a ball, it should touch all round. We therefore prepare a steel tube made quite round and ground flat on the end: the inner edge of this tube becomes a cutting edge; its diameter should be from one half to three quarters of the diameter of the ball. On slipping this tool over the surface say of an ivory ball, the projecting parts of the ivory are scraped away; so that by changing the position frequently, we render the ball truly spherical. A brass tube applied to a lump of stucco in this way soon makes it truly round.

The turning lathe affords facility for rapid work, but must be managed with great care, so as to avoid dwelling unduly on any particular part of the sphere.

In the very same way we grind portions of spheres, as in the manufacture of lenses. This matter will be fully considered hereafter.

When the diameter of the sphere is very great, the surface of a small portion of it, such as that of a lens, becomes flat, so that we may regard a plane surface as the limit of spherical surfaces of exceedingly great diameters. In attempting to prepare a truly flat surface by grinding we are met by this difficulty that the perfect fitting

two does not show either of them to be flat; the one may be hollow or *concave*, the other round or *convex*. Let us then take a third C, and grind it with the first A; if A be convex, C becomes concave, but B was also concave, and therefore B and C will only touch at the margins; hence it follows that if three surfaces A, B, C fit each other, all of them are flat.

The process of grinding is so slow, particularly when metals are used, that we only employ it for finishing; other and more expeditious processes being used for approximation. The grinding also is liable to a serious objection that the emery powder tends to gather in the middle, thus making all the three surfaces slightly hollow, as is seen when we wash away the coarser emery to replace it with finer. It is thus an exceedingly difficult matter to produce surfaces such that their deviation from flatness cannot be detected. There are other difficulties in the way, arising from the fact that all substances are liable to change of shape by pressure, and that the mere laying of the substance on an uneven table may make a perceptible change in the flatness of its surface.

Every student who intends to practise any of the geometrical arts, should prepare for himself a set of flat surfaces, or *try-plates* as they are called. For opticians and instrument makers three good-sized lumps of water sett-stone are very convenient; the young engineer should make his in cast iron.

Any cylinder, whether right or oblique, may be slid in a direction parallel to its axis; but, in addition to this, a *right* cylinder may be turned in its place; so that a cylindric surface may be slid in, as it were, two directions. It cannot be turned in all directions as could the sphere. The cylinder is greatly used in mechanical arrangements,

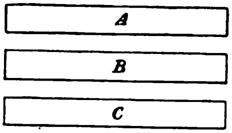
which sometimes require extreme precision. It is not difficult to form cylinders very nearly true; we can do this on the turning lathe, particularly when there is a slide rest; the tinsmith hammers up the tube for a pitcher with great nicety; and we can easily make a paper tube round by pasting one layer upon another; but it is by no means easy to make a cylinder so truly that it shall fit into its place and slide, remaining steam-tight or air-tight all the while. We cannot grind a cylindric plug into a cylindric hole, because by grinding, the former becomes smaller, the latter becomes wider; we can only grind the surface of a cylinder by applying to it a hollowed piece going not quite half round. For small cylinders of brass, such as the eye-pieces of telescopes, we may use pumice-stone having a hollow formed in it; by moving the cylinder lengthways in this hollow and also turning it round from time to time, both the surface of the pumice and the surface of the brass are made true: for finishing, a similar hollow worked in any softish stone, such as the water sett-stone, may be used.

A conical surface is self-gliding but only in one direction, and therefore we cannot grind a cone to be true. We may grind a plug seeming to be conical into a socket so as to make the joint water or air tight; but, as the same line is passed over at each turn, a particle of emery does not change its place but makes a rut all round. Hence it is that unless very great care be taken, conical plugs are not water or air tight without grease to fill up the scratches and small unevennesses, because when the grinding powder is washed away, the plug enters farther into the socket, and a new set of contacts take place.

LESSON XXXIII.

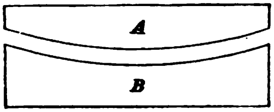
ON SLIDING LINES.

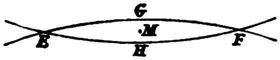
Two straight edges may be slid along each other while their surfaces, so far as they extend, remain in contact; hence we may grind such edges together to make their fitting perfect. But the fitting of two edges A and B does not show them to be straight, it merely shows that they are *even* or *self-gliding*. Now a circular arc is also self-gliding, so the one of them as A may be round, the other B being hollow; and hence, as in the case of plane surfaces, we must have recourse to the mode of triple fitting. We prepare three pieces A, B, C of suitable material and of one size, fit A on B, A on C, and B on C, first by filing or scraping, and afterwards by grinding until all three agree. The student may procure three slips of Welsh slate, or of plateglass, and prepare a set of straight edges for himself; these will serve afterwards to test his working rules. In the case of glass it is advisable to coat the flat surfaces with thick lac varnish to prevent scratches on them; this may afterwards be removed by means of alcohol or of caustic alkali. A set of such standard straight edges is valuable and deserves a neat box for preservation; no artizan should ever purchase such finished, but should finish them for his own use; brass or steel is to be preferred.



For the preparation of lenses and spirit-levels, and for other purposes, we need rules made truly circular, often parts of circles of very great diameter. Now if we grind *two edges* A and B to fit, reversing them end for end and

face for face, to expedite matters, they become circular; we must see that they have the desired radius of curvature. For this purpose we draw, on paper, a line along the edge of one of them; reverse the rule, either end for end or





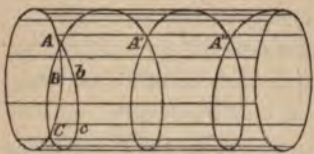
face for face, and draw a new line crossing the former near to the ends as in the figure; we then measure carefully the distance between the cusps E and F and also, but with much greater care, the extreme width G H. These measurements enable us to compute the radius of curvature, by help of the theorem explained in Lesson LIII., Part I. Thus if M be the middle of G H and also of E F, and if the line G H were continued to meet the far circumference in I, the rectangle E M . M F is equivalent to G M . M I so that M I may be computed, and thence G I the diameter of the circle. In the case of very flat arcs a small error in the measurement of G H causes a great error in the diameter; hence we use a very finely divided scale and a magnifying glass when measuring G H.

In order to obtain a greater width at the middle we may extend the arcs by sliding alternately the one rule against the other.

The straight line and the circular arc are the only kinds of sliding lines that can be drawn on a plane surface; and there is only one kind of self-gliding line not all in one plane. If we suppose a cylinder to be turned steadily round while it is also being pushed steadily forward, a tracing point held against its surface will thereon trace a spiral line or *screw*, also sometimes

called a *helix*, but the Greek word ἕλιξ has many meanings, while the English word *screw* has hardly any other meaning than the one.

Let the circumference of the base of the cylinder be divided into equal parts A B, B C, &c., and let lines be drawn on the cylindric surface parallel to the axis: on these lines let equidifferent distances B *b*, C *c*, &c. be marked off; then the points A *b c* . . . indicate the course of the screw. When, having gone round the circumference, we come again to the point A and measure A A′ of the appropriate length, we have made one turn of the screw, and the distance A A′ is called its *pitch*. The second turn is an exact copy of the first, and the process may be continued to any length.



From the very mode of construction we see that each portion of the screw line would fit upon any other portion of it of the same length. The screw of the workshop, however, is not a line, but a *thread* of considerable thickness; sometimes rounded as in the common corkscrew, more commonly angular; all its angular lines being screws of one pitch although belonging to cylinders of different diameters.

Screws may be either *right* (dextral) or *left* (sinistral), the *left* or *left-handed* screw being twin or symmetric to the right-handed one. When a screw on being turned from left to right advances from us we call it *right*, if it come towards us we call it *left*. We shall have again and again to consider the properties of screws, because they are used in machinery for very many purposes.

Beyond these three; the straight line; the circular arc

and the screw; there is no other kind of self-gliding line; so that if two linear pieces of material be ground together until they fit, they must belong to one of these three classes.

LESSON XXXIV.

ON WEIGHT.

Having now seen how to measure and compute the bulks of solids we proceed to consider their *weights*. All the solid and liquid substances with which we are acquainted tend to go downwards or to *fall*; they continue to go down until they meet some obstacle, and, endeavouring still to go down, they *press* against that obstacle. If the hand be used to prevent the fall we are conscious of a strain upon the muscles, and we judge thereby, roughly, of the *weight*. It is at once noticed that the larger bodies are the heavier if they be of one material; but that different materials differ in weight for the same bulk; a piece of cork, for instance, is light; a piece of marble of the same size is much heavier.

The first step towards getting any accurate knowledge on this subject is to seek for some means of *measuring* weight. If we take first one stone and then another in the hand, we easily judge of which is the heavier if there be a great difference, but when they are nearly alike we find it very difficult to estimate between them. Not only so, we are stronger at one time than at another; what we can hardly lift in the morning, is easily carried in the afternoon; we must contrive some means of weighing which shall not be liable to such changes.

The oldest and still the most exact way of estimating

the equality of weights is by the *balance*, which is a rod turning freely on its middle point while the substances to be compared are hung from the two ends. In well-made balances the centre turns on a knife-edge, and each of the pans for holding the object to be weighed or the weight hangs also from a knife-edge. When the distances are exactly alike we expect that equal weights should balance each other; but if the arms be not of equal lengths we find that a lighter mass at the long arm, balances a heavier at the short arm; hence we can at all times verify the beam. If, having adjusted two substances A and B to balance each other on the beam, we reverse their positions, they will not now balance unless the arms have been exactly alike.

As in the case of linear measure, so also in the case of weights, we must assume some standard or unit of comparison. In this country the standard of weight is a brass pound kept in London, which pound is divided into 7000 grains, and also into 16 ounces; the ounce thus containing $437\frac{1}{2}$ grains. By preparing weights of 1, 2, 3, 4; 10, 20, 30, 40; 100, 200, 300, 400 grains, and so on; or weights of 1, 2, 4, 8 ounces, 1, 2, 3, 4 pounds and so on, we are able to ascertain the weight of any object which the balance can safely carry.

Since equal bulks of different materials do not weigh alike, it becomes desirable to examine into the difference; and this is done by contrasting the weight with the bulk. Thus a piece of lignum vitæ measuring 23 cubic inches was found to weigh 7680 grains, so that 1 cubic inch of this wood weighs 334 grains; while a piece of cast iron measuring $4\frac{1}{2}$ cubic inches weighed 8172 grains, or at the rate of 1816 grains per cubic inch. Thus we see that cast iron is more than five times as heavy as lignum vitæ, bulk *for bulk.*

By taking carefully the dimensions of any substance having a definite shape, we may ascertain its bulk; and by weighing it we have the means of computing the weight of one cubic inch of the material. The student should seek for opportunities of performing these operations and should note the results in his pocket-book.

LESSON XXXV.

ON SPECIFIC GRAVITY.

In comparing the weights of different substances it would be exceedingly inconvenient to take them two and two as above, on account of the great number of such combinations; it is preferable to take one substance as the standard of comparison. To an engineer or builder it is useful to know the weight of a cubic foot or of a cubic inch of the materials he works with; but this mode of considering the matter is inconvenient in this much that different nations use different weights and different measures; so that a list of the comparative weights would not serve for all countries.

The substance that has been fixed on by common consent to serve as the standard of comparison is pure water, because it is easily procurable. The specific gravity of a substance is thus its weight as compared with the weight of the same bulk of water.

In order to find the specific gravity of a fluid we ascertain the weight of as much of it as fills a bottle, as also the weight of the fill of the same bottle with water, and we compare the two. It is troublesome to make the computations when the weights are in ounces and quarters;

grains or any other weights counted in tens are more convenient.

Let us suppose the weight of the empty bottle to be 713 grains, and the total weight when full of water to be 2387 grains; there remains for the weight of the water alone 1674 grains. Having put out the water and dried the bottle carefully we fill it with the liquid, say with oil of vitriol (sulphuric acid) and find the total weight now to be 3720 grains, giving 3003 grains for the weight of the acid. Thus we have found that the weight of water is to that of the same bulk of acid in the ratio of 1674 to 3003; hence if the specific gravity of water be put as unit, we have the proportion 1674 : 3003 :: 1·000 : 1·794, and the specific gravity of this sample of sulphuric acid is 1·794. All the sulphuric acid of commerce contains water, some more, some less; and knowing the specific gravity, the bleacher or dyer can discover the proportions of water and acid and so regulate his operations.

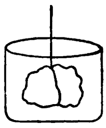
Each kind of oil has its own particular specific gravity, so that, by merely weighing, we may detect a spurious or adulterated oil, excepting when a cheap and a dear oil happen to be nearly alike in this respect.

By preserving the bottle and keeping a note of its weights when empty and when filled with water, we save time in weighing. For the purpose, also, of rendering the computation easy, bottles are prepared to hold exactly 1000 grains of water, in which case the weight of the other liquid needed to fill them gives the specific gravity directly.

Unless the surfaces of a mass of any material be truly flattened, it is almost impossible to determine its bulk by

actual linear measurement, and in order to determine the specific gravity of an irregular lump, the most obvious operation is that used by Archimedes of Syracuse. If we plunge a solid body into a vessel completely filled with water, we shall cause to flow out of it water to the extent of the bulk of the solid; wherefore, if we weigh the overflow, and also the solid, we shall have the data needed for our purpose. This process, however, is not susceptible of precision on account of the vessel necessarily having a wide mouth.

If a jar filled with water up to the brim be weighed, if some of the water be now removed, the solid to be examined placed in the jar, and water be again filled in up to the brim (or to the same mark as before), the total weight will now be the former plus the weight of the solid, less that of the displaced water. Wherefore it follows that the solid when placed in water weighs less than when out of the water by the weight of as much water as it displaces. Let us then suspend the solid from one pan of the balance by a thread so fine that its weight may be neglected, and weigh it in the air. This done, let us bring under it a jar containing water and raise the jar until the solid be amongst the water; it then loses weight, and the difference between its weight under the two circumstances is the weight of the same bulk of water.



As an example of this process we weigh a lump of white marble and find its weight to be 10579 grains; bringing up to it a vessel with water, carefully brushing away any air-bubbles which may adhere to it, we weigh the marble in water and find 6854 grains. The loss of weight here is 3725 grains, wherefore the weights of

equal bulks of water and of marble are in the ratio of 3725 to 10579, and we have the proportion

$$3725 : 10579 :: 1 \cdot 000 : 2 \cdot 840$$

giving 2·840 for the specific gravity of this piece of marble.

N.B. We must here caution the student that this computation does not give us the *real* specific gravity of any substance, it only gives the *apparent* specific gravity, being that which is recorded in almost all the catalogues of substances and in works on chemistry. To get the true specific gravity or that which should alone be used in accurate research, we have to make allowance for the influence of the air. This we shall examine hereafter.

The operation of finding the specific gravity of the materials with which he may have to operate, should be practised by every engineer; it is exceedingly simple, and requires only the use of an ordinarily good balance with small weights. At the end of the volume a table is given of the specific gravities of substances in common use.

LESSON XXXVI.

ON THE COMPUTATION OF WEIGHTS.

A TABLE of specific gravities shows the relative weights of the various substances. In order thence to obtain the actual weight per cubic inch or per cubic foot, it is necessary to determine by actual experiment the weight of a cubic inch of water. This has been done very carefully, and the result, as declared by the Act of Parliament *establishing* the weights and measures, is 252·458

grains; now seven grains make the thousandth part of a pound, wherefore the weight of a cubic inch of water expressed in decimal parts of a pound is ·0360654; from this we compute the weight of a cubic foot of water to be 62·321 pounds.

In order to find the weight of a cubic inch or of a cubic foot of any substance we must multiply the above numbers by the specific gravity; thus platinum having the specific gravity 22·07 is rather more than twenty-two times as heavy as water, and a cubic foot of it would weigh 1375 pounds.

When the dimensions of a solid body are given we are now able to compute its weight, by multiplying the number of cubic units by the weight of a cubic unit of water, and again by the specific gravity of the substance; or, when we have to make several calculations concerning one material, we may rather multiply the weight of a cubic unit of water by the specific gravity and thereby obtain the weight of a cubic unit of the material.

Exercise 1.

Required the weight of a block of marble 6 feet 7 inches long, 4 feet 2 inches broad by 3 feet 10 inches thick, in pounds and in tons.

Exercise 2.

What is the weight of water in a cistern 43 by 32 by 23 inches?

Exercise 3.

Required the weight of a bar of steel $1\frac{1}{8}$ inch square *and 3 feet 6 inches* long.

EXERCISE 4.

From each corner of a cube whose side is 3·4 inches, one inch is marked along each side and the tetrahedrons so indicated are cut off; what is the weight of the remainder made in brass?

EXERCISE 5.

A piece of lead is in the form of a pyramid on a square base whose side is 5 inches, the height of the pyramid being 6 inches; required the weight.

In computing the weights of round bodies we often take the *cylindric inch*, that is the bulk of a cylinder 1 inch in diameter and 1 inch long as the unit of bulk, as thereby we save the frequent multiplication by 3·1415926. The weight of a cylindric inch of water is $\frac{1}{4}\pi \times 252{\cdot}458$ grains $= 198{\cdot}280$. Now the surface of a circle 3 inches in diameter is nine times, the surface of a circle 4 inches in diameter is sixteen times, that of a circle 1 inch in diameter, or in general if the diameter be D inches, the area of the base is D^2 circular inches, and thus the solidity of a cylinder measured in cylindric inches is $D^2 H$.

EXERCISE 6.

What is the weight of a round rod of iron $1\frac{3}{4}$ inch in diameter and 11 feet long?

EXERCISE 7.

Required the weight of a silver disc 1·16 inch in diameter and ·063 thick.

EXERCISE 8.

What is the weight of a gold wire ·0163 thick and *18·27 inches* long?

EXERCISE 9.

A cylindric pitcher is 4·37 inches in diameter and 5·23 deep, what weight of water does it hold?

EXERCISE 10.

The lead weight for a pendulum is 2·51 inches in diameter and 4·85 long; the hole for the pendulum rod is ·36 in diameter; how many ounces of lead are there?

Again, when making computations concerning spheres, we use the *spherical inch*, that is the bulk of a sphere 1 inch in diameter, as the temporary unit of bulk, in which case D^3 is the number of such units contained in the bulk of a sphere D inches in diameter. The sphere is two-thirds of the containing cylinder, wherefore the weight of a spherical inch of water is 132·187 grains.

EXERCISE 11.

A granite ball is 13·29 inches in diameter; required its weight.

EXERCISE 12.

Required the weight of a cast-iron ball 7 inches in diameter.

EXERCISE 13.

Required the weight of a leaden bullet ·427 inch in diameter.

As we discover the weight from the known bulk, so we can determine the bulk when the weight is known. Thus if we wish to know the volume of an irregular mass, the measurement of which geometrically might be impossible,

we may weight it in air and in water, and so ascertain the weight of its bulk of water; then allowing at the rate of 252·458 grains per cubic inch we obtain the volume.

EXERCISE 14.

Required the bulk, in cubic inches, of a stone which lost weight on being placed in water, to the extent of $7\frac{3}{4}$ ounces.

Similarly we are able to compute the dimensions of a body which shall have a proposed weight. Thus we may propose to construct a cylindric measure to hold one gallon of water; that is 10 pounds or 70000 grains. A cylindric inch of water weighs 198·28 grains, wherefore a gallon measure must contain 353·031 cylindric inches, and if D be the diameter and H the height of the cylindric gallon measure we must have

$$D^2 H = 353 \cdot 031 .$$

If the diameter be given we may compute the height by the formula

$$H = \frac{353 \cdot 031}{D^2} ,$$

or if the height be prescribed the diameter is

$$D = \sqrt{\left\{ \frac{353 \cdot 031}{H} \right\}} .$$

And if it were desired to make such a measure with the height equal to the diameter we should have

$$H = D = \sqrt[3]{\{ 353 \cdot 031 \}} .$$

Again the question may arise, "What must be the diameter of a cast-iron ball that shall weigh so many pounds?"

In such a case we have the formula

$$W = \tfrac{1}{6}\pi D^3 \times 252 \cdot 458 \times \text{sp. gr.},$$

whence

$$D = \sqrt[3]{\left\{\frac{6\,W}{\pi \times 252 \cdot 458 \times \text{sp. gr.}}\right\}}$$

when the weight is taken in grains and the diameter in inches.

Exercise 15.

A cylinder of cast iron is to be 13 inches long, and to weigh 50 pounds; what must be its diameter?

Exercise 16.

What length of a copper rod ·732 inch in diameter weighs 12 pounds?

Exercise 17.

An ounce of silver is drawn into wire, which measures 317 feet 6 inches; required the thickness of the wire.

Exercise 18.

In order to get the diameter of a thermometer tube some mercury was put into it and the length thereof measured. The weight of the mercury was found to be 17·31 grains and the length 3·76 inches; required the diameter of the tube.

N.B. In such measurements we have to consider that the quicksilver is curved at the two ends: in order to avoid the error caused by this curvature we first introduce a small quantity of mercury, weigh and measure, then introduce more, again weigh and measure, take the differences and compute with these.

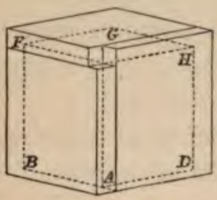
Here we see that, for business purposes, we must be able to extract the cube root of a given number. Now we

shall hardly be able to extract cube roots without knowing something about cube numbers, therefore we shall proceed to study these.

LESSON XXXVII.

ON CUBE NUMBERS.

If we make a list of the successive cube numbers and take the differences, the differences of these differences or the *second* differences; and the differences of these again we find the third difference to be 6 all along; so that we may continue the table of cubes by addition only, just as we did with the table of squares. In order to see clearly that this third difference must ever continue to be 6, we shall suppose the figure A B C D E F G H to represent a cube on the side A B = 7 units, and therefore containing 343 cubic units, and add to it enough to make up the cube of 8 units. For this we shall place three plates each 1 inch thick upon the three faces A B F E, E F G H and H D A E, each plate being 7 units broad and 8 units long so as to project 1 unit on each side. In this way we build up the cube of 8 units, with the exception of a cubic unit at the corner E. Thus we see that the cube of 8 exceeds the cube of 7 by thrice the product of 7 by 8, with unit more, or that



$$8^3 - 7^3 = 3.7.8 + 1$$

and thus adding 168 + 1 or 169 to 343 we get 512 which is just the cube of 8. In the same way the difference between the cubes of 8 and 9 is $3.8.9 + 1 = 217$; and so on. Hence the second difference must be the difference between thrice 7.8 and thrice 8.9; now the difference

between 7 times 8 and 9 times 8, is twice 8, wherefore the second difference is three times that, or 6 times 8, 48.

The next of the second differences being, in the same way, six times 9 or 54, the difference of these, which is the third difference must be 6 and must continue to be 6.

Hence a list of cube numbers is easily carried forward; thus to begin with the cube of 60 which is 216 000, we find the cube of 61 by adding $3.60.61+1=10981$, making 226 981. The second difference, which falls to be added to the above 10 981 is six times 61 or 366, and the third difference is 6.

If then we write these numbers 216 000, 10 981, 366 and 6 in a line as shown in the subjoined example, and form the next line by adding to the number in each column, that on the same line in the column immediately to the right, we obtain a table of cube numbers. With a little practice a moderately good computer may learn to write in the sums beginning from the left hand all along.

When the work has been carried so far as to the cube of 70 we may test the work by actual multiplication or by reference to an earlier part of the table.

No.	Cube.	1st Diff.	2nd Diff.	3rd.
60	216 000	10 981	366	6
61	226 981	11 347	372	6
62	238 328	11 719	378	6
63	250 047	12 097	384	6
64	262 144	12 481	390	6
65	274 625	12 871	396	6
66	287 496	13 267	402	6
67	300 763	13 669	408	6
68	314 432	14 077	414	6
69	328 509	14 491	420	6
70	343 000	14 911	426	6
71	357 911	15 337	432	6
etc.	etc.	etc.	etc.	

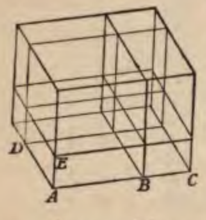
The student may continue this table as far as to the cube of 100. If he have already a table of cubes, he may extend

it by some hundreds as an exercise; he should practise addition from the left hand in doing this.

In seeking the cube root of a number we can only proceed by trials. Having made a guess, we try whether the guess be right, by actually cubing the supposed root; taking notice of the error, if there be one, we make a second guess, cube the newly assumed root, examine the error, and so proceed until we obtain the root either exactly or with precision sufficient for our purpose.

The labour of these operations is often great, and becomes more and more so as we extend the exactitude to several decimal places; and we search for some shortening of the work. Putting A for our first guess we cube it and get A^3; we then make another guess B and proceed to cube B; but B is not much different from A, so it may be better not to cube B directly but to calculate the difference between the two cubes. In order, therefore, to prepare ourselves for extracting a cube root rapidly, we must learn how to pass from the cube of one number A to that of another number B; just as we had to do in learning to extract the square root.

If we construct a cube on A C, the sum of A B and B C, divide each of the sides into corresponding parts as at D and E, and through the points of section B, D, E, lead three planes parallel to the faces of the cube, we divide it into eight portions. Among these we find, placed at opposite corners, the cubes of A B and of B C. In contact with the cube of A B there are three oblongs each on the square of A B and having B C for its thickness; and in contact with the cube of B C there are three other oblongs each on the square of



B C and having A B for its length. Thus, writing shortly, $AC^3 = AB^3 + 3.AB^2.BC + 3.AB.BC^2 + BC^3$; the same law holds good of numbers, so that if A be the assumed root and e the correction applied to it we have

$$(A + e)^3 = A^3 + 3.A^2.e + 3A.e^2 + e^3,$$

and therefore the addition to be made to the first cube in order to get the new cube is

$$3A^2.e + 3A.e^2 + e^3;$$

now we have A^2 ready and, as e is small, the computation of e^3 is not laborious, so that the computation of this addition is preferable to the direct calculation of $(A + e)^3$.

Let us look forward another step; writing B instead of $A + e$, let f, a second correction, be applied to B, the addition to be made to the cube of B is

$$3B^2.f + 3B.f^2 + f^3,$$

but since we have not got B^2, the second correction to the cube is not so easily applied, and thus we must contrive some means for shortening the labour of computing the square of B; or better still of computing $3B^2$ from $3A^2$. While extracting the square root we found B^2 from A^2 by help of $2A$, wherefore we shall now deduce $3B^2$ from $3A^2$ by help of $6A$. Hence if we form a set of columns headed with 6, $6A$, $3A^2$ and A^3 as in the accompanying scheme, multiply each of the three 6, $6A$, $3A^2$ by the correction e, writing the product in the column to the right; multiply then the products $6e$, $6Ae$ by $\frac{1}{2}e$ moving them again one step to the right, and lastly multiply the $3e^2$ by $\frac{1}{3}e$, placing the product e^3 in the column headed A^3, we obtain by summation, the values of 6, $6B$, $3B^2$ and B^3 and are then ready to proceed with the second correction f.

6	$6A$ $6e$	$3A^2$ $6A.e$ $3.e^2$	A^3 $3A^2.e$ $3A.e^2$ e^3	A $+e$
6	$6B$ $6f$	$3B^2$ $6B.f$ $3.f^2$	B^3 $3B^2.f$ $3B.f^2$ f^3	B $+f$
6	$6C$ &c.	$3C^2$ &c.	C^3 &c.	C

In order to see the application of this process we shall build up the cube of 47836, fancying that the 7 was only thought of after the 40000 had been cubed, that the 8 (or rather 800) was not known till the previous 47000 had been cubed, and so on.

6	6 A	3 A^2	A^3	A
6	240 000 42 000	4 800 000 000 1 680 000 000 147 000 000	64 000 000 000 000 33 600 000 000 000 5 880 000 000 000 343 000 000 000	40 000 7 000
6	282 000 4 800	6 627 000 000 225 600 000 1 920 000	103 823 000 000 000 5 301 600 000 000 90 240 000 000 512 600 000	47 000 800
6	286 800 180	6 854 520 000 8 604 000 2 700	109 215 352 000 000 205 635 600 000 129 060 000 27 000	47 800 30
6	286 980 36	6 863 126 700 1 721 880 108	109 421 116 687 000 41 178 760 200 5 165 640 216	47 830 6
6	287 016	6 864 848 688	109 462 300 613 056	47 836

Here we are supposed to make a first trial with A = 40 000; this gives $A^3 = 64\,000\,000\,000\,000$, which is

say too small for our purpose; we therefore apply a correction $e = 7000$, making $B = A + e = 47\,000$. In order to build up the cube of B, and also to prepare for subsequent expected corrections, we compute $6\,A$ and $3\,A^2$, writing them at the heads of their respective columns. Multiplying 6, $6\,A$ and $3\,A^2$ by e or 7000 we write the products in the adjoining columns and so form the second line. The numbers in the second line (with the exception of that in the column A^3) are multiplied by $e = 7000$, and the products halved to form the third line; lastly the third part of $3\,e^2$ (147 000 000) is multiplied by 7000 to get the cube of e which is the only number in the fourth line.

The sums of all the numbers found in the several columns give the values of B, B^3, $3\,B^2$, $6\,B$ and 6 wherewith to proceed in the same way for the second correction, in this case supposed to be 800; and so the work is continued until we arrive at the cube of 47836.

In the above scheme all the cyphers have been written, but in actual work we omit those which serve merely to determine the positions of the figures, taking care, however, to preserve the positions as if the cyphers were written.

The student may practise this operation on any number assumed by him; or one student may propose to another the highest digit, then the second and so on; each proposition being made only after the previous computation has been finished.

LESSON XXXVIII.

ON CUBE ROOTS.

THE extraction of the cube root of a number
easy by the preceding lesson. Our first busi
make a guess, and this guess need not be a v

we observe that the cube of say 40 is just one thousand times the cube of 4; wherefore if we mark off groups of three figures each, counting from the units' place, we shall know the number of digits in the root, and may easily know the highest digit, since it is not difficult to keep in mind the cubes of the nine digits.

Thus if we wish the cube root of the number 502 973 801 627 we observe that it consists of four groups of three figures and that 502, the highest group, is between 343 the cube of 7, and 512 the cube of 8, wherefore the required cube root must be between 7000 and 8000, and evidently nearer to the latter. Making then our first trial with 7000, we find its cube to be 343 000 000 000; and on deducting this from the given number we obtain the error 159 973 801 627, and we have to judge of what addition should be made to the first guess 7000 in order to cause an augmentation of the cube to this extent. Now if we add some correction which we shall call e to 7000 the cube will be increased by the product of 3×7000^2 by e and something more; wherefore by dividing the above error by 147 000 000, we shall get somewhat more than the value of e; now the quotient is 1000, and we know already that 1000 is too much; so we may try the correction 900. The effect of this correction will be to augment the cube and thereby to diminish the error; computing the augmentation of the cube in the manner just explained we subtract that augmentation from the first error and leave only 9 934 801 627 for the error of the cube of 7900, as is seen in the subjoined scheme. Dividing this new error by thrice the square of 7900 we get about 50 for the second correction which is to be applied in the same way.

			502 973 801 627 343	
6	42 000 5 400	147 000 000 37 8 2 43	159 973 801 627 132 3 17 01 729	7 000 900
6	47 400 30	187 230 000 2 370 7 5	9 934 801 627 9 361 5 59 25 125	7 900 50
6	47 700 12	189 607 500 95 40 12	513 926 627 379 215 0 95 40 8	7 950 2
6	47 712 4	189 702 912 33 398 1	134 616 219 132 792 038 11 689	7 952· ·7
6	47 716·	189 736 311·	1 812 492·	7 952·709 553

In this way we find the cube root of the proposed number to exceed 7952, there being an error of 134 616 219. The extraction may now be continued to fractional parts of the unit. Unless, however, an extreme degree of precision be demanded, it is not necessary, in such a case as this, to carry out the remainders to fractional places; and after the first step it is seen that several of the subsequent figures of the root may be got by division alone.

This process differs somewhat in its arrangement from that usually given in books on arithmetic; the advantage of it lies in this, that the very same arrangement is useful in a great many much more difficult problems, to which the usual process cannot be extended.

Exercise 1.

Extract the cube root of 4686533117023459 7376.

Exercise 2.

Extract the cube root of 2718281·828459 .

Exercise 3.

Extract the cube root of 2000000 .

When we have to extract the root of a fraction, we must be careful to observe that its value is always between unit and the fraction. Thus the cube of half an inch is the eighth part of a cubic inch, that is to say $\frac{1}{2}$ is the cube root of $\frac{1}{8}$; in the same way $\frac{1}{10}$ is the cube root of $\frac{1}{1000}$. Hence if the cube root of ·037 826 517 were required we should divide it in groups of three figures from the units' place; the highest group is ·037 or $\frac{37}{1000}$, the cube root of which is between $\frac{3}{10}$ and $\frac{4}{10}$, so that the root sought for begins with ·3 and the work proceeds as shown below.

			·037 826 517 ·027	
6	1·8 ·18	·27 54 2 7	10 826 8 1 81 27	·3 ·03
6	1·98 30	·326 7 9 90 75	1 889 517 1 633 5 24 75 125	·33 5
6	2·010 3 6	·336 675 1 206 1	231 142 202 005 362	·335 6
6	2·013 6 5	·337 882 161	28 775 27 030 6	·335 6 8
6	2·014 1	·338 043 1	1 739 1 738	·335 68 5 14

When an extensive table of cube numbers is at hand the calculation may be much shortened; thus in extracting the cube root of the number $\pi = 3 \cdot 14159\ 26536$ we find the first four digits, from any table of cubes, to be $1 \cdot 464$, three times the square of which is $6 \cdot 429\ 888$, whence the operation

			3·14159 26536 3·13778 5344	
6	8·784 30	6·42988 8 439 20 7	380 73096 321 49440 10980 1	1·464 5
6	8·7870 5	6·43428 07 79 08	59 12675 57 90853 356	1·4645 9
6	8·7875	6·43507 15 88	1 21466 64351	1·46459 1
6	8·7875	6·43508 03 78	57115 57117	1·46459 1 8876

EXERCISE 4.

Required the side of a cube of granite which shall weigh exactly one ton.

EXERCISE 5.

To construct a regular octahedron in lead which shall weigh 112 pounds.

EXERCISE 6.

What is the diameter of a ball of ivory weighing 3 ounces?

EXERCISE 7.

Required the dimensions of a cylindric measure having its depth equal to its diameter and holding one gallon (70000 grains) of water.

Exercise 8.

What is the diameter of a sphere, and what is the side of a cube of platinum each of which shall weigh one grain?

Exercise 9.

Construct a cone of brass having its height equal to its diameter and weighing one pound.

Exercise 10.

Required the diameters of spheres of brass weighing 1, 2, 3, 4, 5, 6, 7, 8 ounces.

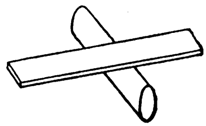
LESSON XXXIX.

ON THE BALANCE WITH UNEQUAL ARMS.

When the arms of a balance are unequal, the weights are also unequal, and it has been known from very ancient times that the weights are inversely proportional to the lengths of the arms; thus a weight of two pounds at the end of an arm of one foot, balances a weight of one pound at the distance of two feet from the fulcrum; or a weight of two pounds at the distance of three feet, balances one of three pounds at the distance of two feet.

Many speculative writers have tried to show that this *ought* to be, or to demonstrate logically that this *must* be the relation between the weights and the lengths, but no one has been successful; on examining their arguments we find that, at some step or other, the very thing to be proved has been taken for granted in another form. Our knowledge of this law is derived from experiment, and

therefore the student should personally examine into its truth. For this a very simple apparatus, copied from the boy's see-saw, is enough; a straight flat even rod is to be marked off in equal part from its middle, and to be laid across a cylinder such as a common round ruler. When the middle is placed over the ruler the rod lies level. If now we place a weight of four ounces at the distance of seven inches on the one end, and a weight of seven ounces at the distance of four inches on the other end, the rod with these weights will still be level, showing that the one weight just balances the other; and in whatever way the trial may be varied the same *inverse* proportion holds good; so that we believe it to be true in those cases which we have not actually tried, and make calculations accordingly, feeling quite sure that the results will prove correct.



An arrangement of this kind is called a lever (from the French, meaning to lift) because in the form of *crowbars* or *pinches*, it is used to lift heavy masses. The point on which the lever turns is called its *fulcrum*.

This is a very good example of the manner in which knowledge is really acquired. We make several trials, experiments or observations, on the particular subject, look among the results for some connection; change the circumstances, see whether the same connection hold good; finding that it does so, we conclude that it is really a law and thereafter we calculate or *predict* by its help.

Thus, in the present case, when three of these four are given, viz. the two arms and the two weights, we calculate beforehand what the fourth must be; just as, from the measurement of two angles of a trigon, we tell what the third is, without measurement.

Exercise 1.

A weight of 37 ounces is placed at the distance of 11 inches on one arm, and a stone placed at 13 inches on the other arm is found to balance it; what is the weight of the stone?

Exercise 2.

A weight of 39 ounces is placed at the distance 19 inches on the one arm, at what distance must a weight of 23 ounces be placed on the other arm in order to balance the former?

Exercise 3.

The total length of a crowbar being 57 inches, and the fulcrum being 5 inches from the thick end; a pressure of 70 pounds is exerted on the longer end in order to budge a stone; what has been the pressure at the shorter end?

Exercise 4.

The arms of a balance are slightly unequal, the one being 5·27 and the other 5·29 inches; a substance was placed in the pan at the short arm and was found to *seem* to weigh 8329 grains; required its true weight; required also its *apparent* weight when placed in the other pan.

By weighing a substance first in one pan and then in the other pan of a balance we obtain the means for calculating its true weight, and also for discovering the ratio of the inequality of the arms. One might think that the substance will seem to be as much too heavy in the one pan as too light in the other; and that therefore the half sum of the two apparent weights should give the true weight. In the case of well-made balances which are as

nearly just as the workman can manage to make them, the half sum of the apparent weights is sufficiently near for almost any purpose; but in the case of balances made intentionally false, we must examine the matter more carefully in order to get at the truth. Let us suppose that an unjust grocer uses a beam whose arms are 5 inches and 6 inches long, putting his weights in the pan at the short arm; and that a customer asks for 30 ounces of sugar. Weighed in this way he will only get 25 ounces. Suspecting unfair dealings, the customer asks a second time for 30 ounces to be weighed in the other pan, and then he obtains 36 ounces; so that altogether he has got 61 ounces instead of 60 ounces for his money. Here we see clearly that the difference between the mean and the greater is more than the difference between the mean and the lesser weight, and that therefore the mean must always be less than half the sum of the two extremes; it is the mean proportional between them or, as it is called without any meaning, their *geometric* mean. Let us take the case of a balance having its arms as two to one; then a substance whose true weight is 10 pounds would be balanced in the one pan by 5 pounds, in the other pan by 20 pounds; the one apparent weight being four times (that is twice twice) the other.

Therefore in order to get the true weight we must take the product of the numbers representing the apparent weights and extract the square root of that product. Thus in the last example the apparent weights were 5 pounds and 20 pounds; the product of 5 by 20 is 100 of which the square root is 10; or in the preceding example the extreme weights were 25 ounces and 36 ounces; the product of 25 by 36 is 900 whose square root is 30.

Knowing the ratio of the mean weight to one of the extremes, we are able to compute from one weighing only, the true weight of any other object as given by the balance.

EXERCISE 5.

In order to determine the error of his balance, an analyzing chemist weighed a body in each of the pans; the apparent weights were 3873 grains and 3875 grains; required the true weight, and also the error by taking the half sum of the extremes.

EXERCISE 6.

The apparent weights being 5281 and 5234 grains, required the true weight; also, the shorter arm being 4·29 inches, required the error of the longer arm.

When we place a weight on one arm, it tends to turn the lever round; a weight placed on the other arm tends to turn the lever in the opposite direction, and when the two tendencies are alike, the lever is balanced. If instead of turning on a fulcrum the lever be attached to an axle, a weight applied to it tends to twist the axle; and therefore we may call this tendency to turn the lever round by the name *torsion* or *twisting*.

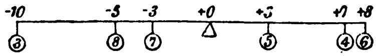
A weight of one pound placed at a distance of one inch causes a certain tendency to twist, or *torsion*, and this we may take as the unit of torsion; a weight of three pounds at the distance of four inches from the axis causes twelve times as much torsion, as much as a weight of twelve pounds at the distance of one inch, or as a weight of one pound at the distance of twelve inches. This product then of the two numbers representing the weight and the length of the arm, expresses the amount of torsion

measured in its own units. For shortness' sake we generally say "the product of the weight by the length of the arm is the torsion," but the student must be careful to note that there is an *ellipsis* in this statement.

This product is often called the *moment*, but this word moment is used in several other senses by writers on mechanics; any word that is to be used in only one sense is preferable. To say then that the torsion (or moment) due to the weight on the one arm is equal to that caused by the weight on the other arm, is only to say that the weights are proportional inversely to the lengths of the arms.

When there are several weights on the one arm and perhaps several also on the other, the torsions caused by the one set must make up as much as those caused by the other set of weights, when the lever is balanced.

Thus if on the right arm of a lever at the distance 8 inches there be a weight of 6 oz.; at 7 inches one of 4 oz.; at 3 inches one of 5 oz.; and if on the left arm there be at 3 inches a weight of 7 oz.; at 5 inches one of 8 oz., and at 10 inches one of 3 oz.; the lever is balanced, for the torsions (or moments) on the right hand are $8 \times 6 = 48$, $7 \times 4 = 28$, $3 \times 5 = 15$, making in all 91; while those on the left hand are $7 \times 3 = 21$, $8 \times 5 = 40$, $3 \times 10 = 30$ making also 91.



It is usual and also very convenient to mark distances measured from the fulcrum on the one side with the sign $+$, those measured on the opposite side with the sign $-$, and to mark the corresponding torsions in the same way, *in which case* we say that the total torsion (having rega

to the signs) is nothing. The above example would then be represented in the subjoined table:

Weight.	Arm.	Torsion.
6	+ 8	+ 48
4	+ 7	+ 28
5	+ 3	+ 15
7	− 3	− 21
8	− 5	− 40
3	− 10	− 30
		0

This balance or *equilibrium* only holds good of the weights; the rod or lever must be balanced independently.

Exercise 1.

Three weights are placed on one arm of a lever; 5 ounces at 3 inches, 6 ounces at 4 inches and 8 ounces at 7 inches from the fulcrum; at what distance must a weight of 19 ounces be placed in order to balance them?

Exercise 2.

On one end of a lever a weight of 11 ounces is placed at 3 inches, one of 7 ounces at 5 inches and one of 8 ounces at 7 inches from the fulcrum: on the other arm 5 ounces at 3 inches, 9 ounces at 6 inches and 7 ounces at 9 inches; where shall we place a weight of 3 ounces in order to make a balance?

Exercise 3.

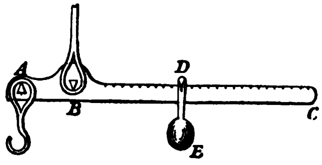
On one arm there are 23 ounces at 11 inches, 37 ounces at 15 inches, 31 ounces at 19 inches; on the other arm 29 ounces at 13 inches, 17 ounces at 17 inches, and 41 ounces at 18 inches; where must a weight of one ounce be placed to produce equilibrium?

LESSON XL.

ON THE STEELYARD.

INSTEAD of weighing bodies by placing different weights into a pan kept at a fixed distance from the fulcrum, we may use a fixed weight shifted to different distances on the arm.

The steelyard is suspended by a pair of rings supporting the two ends of a knife-edge B, which forms the fulcrum. The object to be weighed is hung to a hook attached to two rings hanging on the ends of a second knife-edge A; the distance A B thus forming the shorter arm. The long arm B C is notched on its upper edge at equal distances the notches being to receive the knife-edged ring D from which the fixed weight E depends.



In order to graduate the long arm we hook on a known weight as small as can be weighed on the steelyard and bring D E close to the fulcrum until it balance that weight, marking the position. We then append a second known weight as heavy as the instrument can receive, shift D E out until the balance again take place, making a second mark. The distance between these marks has then to be divided to suit the difference between the two known weights, and the divisions have to be numbered accordingly.

In this case the lever is not balanced, so that the weight of the arm has to be allowed for; in the above mode of making the graduations this allowance is made. This may also be done by trying what weight is balanced by the long arm without the weight E and its ring D.

EXERCISE.

The movable weight of a steelyard is 15 ounces; when placed near to the fulcrum it balances 3 pounds; when placed at the end it balances 15 pounds, and the distance between the two positions is 18 inches; what must be the lengths of the graduations for pounds and ounces? also what is the distance from the fulcrum to the fixed knife-edge?

LESSON XLI.

ON PRESSURE.

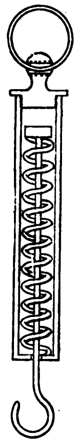
WHEN a weight is hindered from going downwards it *presses* against the obstacle and changes the shape thereof. The change of shape is often unnoticed because of its smallness but is not therefore the less real. Some substances are more changed than others; thus if we place a piece of Indiarubber on the table and put a weight on the top of it, the rubber is seen to be squeezed thinner and forced out at the edges; but the same weight may make no perceptible change on a piece of hard wood.

The change of shape is most easily noticed in the bending of rods. Thus if a plank be laid upon props at the two ends and if a load be placed on the middle, the plank is seen to be bent; when the weight is removed the plank returns to its former shape. A thin plank is much more easily bent than a thick one. Some substances take a *sett* when sufficiently bent; thus a piece of iron wire if slightly bent returns to the same shape when the pressure is removed, but if much bent it returns a little, but, roughly speaking, retains the new shape. Lead wire can hardly be bent at all without a *sett*; whereas other substances as

hard steel and glass do not take a sett but break over. These latter substances are said to be *hard*, the others are *soft*, *ductile*, *malleable;* and often the same substance may be soft in one condition and hard or *brittle* in another. Thus glass when red hot is quite soft, when cold it becomes brittle; whereas brass, which is moderately soft when cold, becomes brittle on being heated.

When a substance returns to the same shape on the removal of the pressure we call it *springy* or *elastic;* and such a substance when so shaped as to be easily bent or changed is called a *spring*. Now the extent of change depends on the amount of pressure, and so may serve to indicate or measure it.

A very convenient arrangement is to coil a steel wire in the form of a screw, leaving a space between the coils, and then to *temper* it. When this is set upon the table and a weight placed on the top, it is compressed, the amount of compression being nearly proportional to the weight; but as this is sure to topple over, it is usual to place the spiral spring in a cylindric box so as to rest on the bottom. Through a hole in the centre, a rod passes along the middle of the coils. On the upper end of this rod there is a cap which rests on the top coil, and to the lower end of the rod a hook is attached, by which to suspend the weight. Graduations are made on the rod to show the weights.



Instead of compressing the spring we might stretch it by simply hooking on the weight to the lower end; indeed the indications in this way are more delicate, because there is no rubbing against the sides of the tube or against the rod; but then if the *stretched spring* be overloaded it is broken or spoiled;

whereas an overloading of the compressed spring only brings the coils together and does no harm. Such spring weighing machines are very useful when great nicety is not needed.

When a weight is placed on the top of a spring, it bends the spring until the elasticity be able to resist the gravitation; but it cannot compress the spring unless there be resistance at the lower end, which end must exert as much pressure against the table as the weight would have done directly. Thus we see that there can be no pressure without resistance; we cannot squeeze anything by pressing only on one side of it; we must press on opposite sides; neither can we stretch a string by pulling one end; there must be something to pull the other end as well.

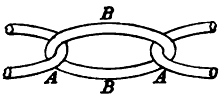
This leads us to puzzling thoughts. The weight presses on the spring; the spring presses on the table-top; the table-top again is really a spring not nearly so flexible as the steel one but acting in the same way, it presses on the legs of the table; these on the floor; the floor presses on the walls; the walls on the ground; and the ground, on what? For the present we may regard the ground as the ultimate resister; when our knowledge is more extended we may come to see otherwise.

Incorrect ideas as to pressure and resistance lead to sad mistakes: a beam is very strong; it will carry a great load—no; it can only do so when the supporting walls are sufficient.

We exert pressure by means of our muscles and limbs; we push; we pull; we twist. On trying with all our strength to push away a heavy mass we find our feet to slip on a smooth floor; without a rough floor or something to resist we cannot exert our strength. Neither can we

pull towards us, or turn a heavy thing round without secure footing; and whatever be the source of the resistance it is as much pushed, pulled or twisted in the one direction as the object on which we are trying to operate is, in the opposite direction. Hence when we are in a carriage or boat, all attempts to push the conveyance forward are in vain, because when with our hands we push one part forwards, we push some other part as much backwards with our feet. He was an ignorant passenger who tried to keep a boat level by setting his shoulder to the mast.

We are apt to speak, or rather to think, of *pulling* carelessly. In point of fact we cannot *pull* any body towards us, we always *push* it; to reach to the body is not enough, we must get our fingers beyond and on the farther side of it; or we must get hold of it by *pressure*. The pulling or stretching is in our own arm. Similarly we cannot support any article by hanging; the hook or string must go over the nail and press on the upper side thereof; and again the hook or tie at the lower end of the string must pass below some part of the article which is to be hung, and *press* it upwards. Neglect of these considerations have led to ridiculous blunders in construction. A substance may be compressed without distension; but no substance can be distended in any part without compression in some other part. We see this quite clearly in the case of the link of a chain; when the chain is stretched each of the inner surfaces at the ends A, A is compressed, while the sides at B and B are stretched. The strain is transmitted from link to link along



the whole chain, each link being compressed at some parts and distended at others.

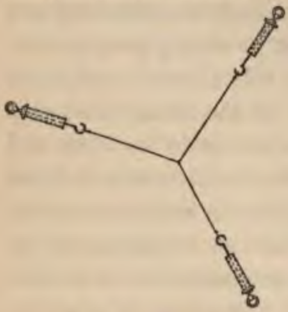
EXERCISE.

A lighter is almost aground near a pier, on board there are five men and on the pier there are seven. A rope is thrown ashore; how should the men be set to work to bring the lighter to the pier as quickly as possible? and how are all the men to be employed?

LESSON XLII.

ON THE BALANCE OF THREE PRESSURES.

IF we knot the ends of three threads together, or if we make them fast to a small ring, and then pull the other ends, the threads arrange themselves to make angles which

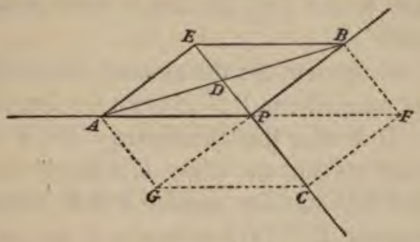


change when the strains are changed. In order to discover the law connecting these angles with the three pressures, we may tie a small spring balance, such as is used by sportsmen, to each of the threads. Arranging this little apparatus on the surface of a table, we may slip a sheet of paper under the threads, and mark on it the directions as well as the intensities of the pressures, so as to get a record of the actual experiment.

The very first case which one would like to try is when the three strains are alike; it is then found that the three angles are alike; indeed we should have expected *this.* *When* the strains are unequal the greater strain is *'ways opposite* the lesser angle; and it is observed that

the direction of any one strain when continued beyond the knot, passes into the angle made by the other two.

In order to show the result of a trial we may measure along each thread a distance proportional to the strain on it, so that the three lines P A, P B, P C may represent



both the directions and the intensities of the three pressures. The student may make a considerable number of trials and record them in this way, putting say one inch of line to stand for one ounce of pressure; or using any other convenient scale.

This being done, if he join two of the points, say A and B, and continue the third line C P to cross A B, he will find the crossing to take place always at the middle; this bisection of the line A B then seems to be the *law* of the matter.

As in the case of the lever, many attempts have been made to discover, by mere logic, the law of the balance of three pressures; but the basis of the arguments is always an assumption of the truth of the very law which is to be found. The only foundation for our knowledge of this law is accurate trial.

This cannot be a true law unless it be consistent with other known laws; now the statement of it involves a purely geometrical theorem, and thus we have an opportunity of examining whether this law of pressures be in accordance *with those of* geometry, which also were purely experimental.

If this statement be a true one, it must hold of the lines B C, and C A as well as of A B; and to state it is virtually to assert that if each of the sides of the trigon A B C be bisected and the middle points joined with the opposite corners, the three joining lines meet in one point P. If this assertion be not true, the law as above stated cannot possibly be so.

Putting out of view, for the moment, the length of the line P C, let us continue P D to twice the length and join A E, E B. It is clear that this figure A E B P is a rhomboid. If now we shift the trigon E P B into the position P C F, and A E P into the position P B F, we make up a second rhomboid P B F C; or again if we produce B P and draw A G parallel to E P, G C is parallel and equal to A P. Now the diagonals B C and C A would be bisected by P F and P G, wherefore the law of bisection, as above stated, is consistent with the geometrical laws.

This theorem may be stated in another way. The pressures represented by P A, P B, P C are parallel and proportional to the three sides P A, A E, E P of the trigon P A E; and thus we may say that "when three pressures exerted against a point balance each other, they are proportional to the three sides of a triangle drawn parallel to their directions"; and we may also say that "when three pressures parallel and proportional to the three sides of a trigon act on one point they balance each other"; and again we may put it in this way, that "if a rhomboid (as P A E B) be formed having the representatives of two pressures as its sides, the diagonal (E P) will represent the third pressure which balances them." These are only different ways of stating the same truth.

This theorem concerning the equilibrium of three pressures, *is the* foundation of all accurate knowledge of

mechanics; it connects mechanics with geometry and with arithmetic, and enables us to compute beforehand the strains to which a proposed structure is liable.

Exercise 1.

Three pressures of 24, 31 and 35 ounces balance each other at a point; construct the scheme of the equilibrium and measure the angles with the protractor.

Exercise 2.

The spring balances show the tensions 5, 7 and 8 ounces; find by construction the angles made by the three threads.

Exercise 3.

The three strings make angles of 114°, 120°, and 126°; the strain on the thread opposite to 114° is 85; find by construction the remaining strains.

Exercise 4.

Pressures of 487 and 437 making an angle of 111° are balanced by a third pressure; required its direction and intensity.

Exercise 5.

Pressures of 252 and 275 at right angles to each other are balanced by a third pressure; required its direction and intensity.

Exercise 6.

Pressures of 88, 105 and 137 balance each other; required their angles.

Exercise 7.

Two pressures of 89 each making an angle of 128° are balanced by a third; required its intensity.

EXERCISE 8.

Two pressures of 149 are balanced by a third pressure of 102; required the angles.

EXERCISE 9.

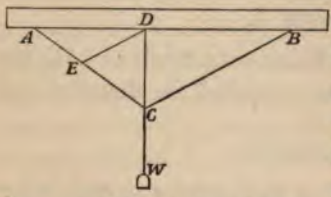
Required the angles of the pressures 145, 145 and 48.

LESSON XLIII.

ON THE COMPUTATION OF STRAINS.

IN order to become acquainted with the application of the above theorem, we may examine a few of the simpler kinds of structure.

Let us suppose that two hooks are fixed, at A and at B, on the under side of a level beam, and let a string A C B be attached to these; also at some point C in that string let a thread be fixed carrying a weight W. The particulars of this arrangement, namely the lengths of AB, BC and CA, and the weight W being known, we may thence compute the tensions of A C and C B.



There are three strains acting at the point C, and in order to get their proportions we must form a trigon having its sides parallel to the three directions; we may do this in various ways. In our figure, W C is shown produced to D, and the line D E is drawn parallel to B C, *thus forming* the trigon C E D. Now we have already *learned* by Lesson XLIX., Part I., how to compute A D,

and D C; wherefore, since A D E is similar to A B C, D E and E C may be found. Knowing now the sides of D C E we know the ratios of the tensions of the three strings C W, C A, C B; wherefore when one of these is known we may compute the others.

As an example let us suppose A B to be 48 inches, and the string to be 64 inches, divided at C into two portions A C = 29 inches C B = 35 inches, and let the weight at W be 36 ounces. We easily find A D to be 20 and D C to be 21 inches, wherefore, from the proportion A B : B C :: A D : D E, we find D E = $\frac{175}{12}$; and again from the proportion A B : B D :: A C : C E, C E comes out $\frac{203}{12}$. Now C D is 21 inches or $\frac{252}{12}$, wherefore the sides of the triangle D C E are in the proportions

$$DC : CE : ED :: 252 : 203 : 175,$$

which must also be the proportions of the three strains. But the strain C W is 36 ounces, wherefore the tension of C A must be 29 ounces, and that of C E, 25 ounces.

The student will here observe that the *proportions* of the sides D C, C E, E D are needed, not the actual lengths, except in so far as these help us to get the proportions. He should carefully work out the accompanying exercises and contrast the results for flat trigons with those for acute angled ones; he should also change the position of the auxiliary trigon, as by drawing through A a line parallel to C B and continuing C D to meet it, or otherwise.

Exercise 1.

A string 33·8 long is attached to two hooks 23·8 apart on the same level, and a weight of 6000 grains is hung on at the middle; required the strain on the string.

Exercise 2.

The distance of the hooks being 11 inches, the length of the string 14·6 and the weight 2400 grains; required the tension when the weight is hung at the middle.

Exercise 3.

The horizontal distance of the hooks being 9 inches, the length of the string 10·6, and the weight hung at the middle one pound (7000 grains); required the tension.

Exercise 4.

The distance being 30 inches, the string 34 inches, and the weight at the middle being 48 ounces; required the strains.

Exercise 5.

The distance of the hooks being 3 feet and the string being 39 inches long, what are the strains caused by a weight of 20 lb. hung at the middle?

Exercise 6.

The hooks being 70 inches apart and the string 74 inches long, what strains are caused by a weight of 60 ounces at the middle?

Exercise 7.

The distance of the hooks is 5 feet 3 inches, the length of the string 5 feet 5 inches, and the weight at the middle 7 lb.; required the strain on the string in ounces.

Exercise 8.

The distance of the hooks is 7 feet, the length of the string 7 feet 1 inch, and the weight at the middle 4 lb. 1 oz.; required the tension.

Exercise 9.

The distance being 15 feet, and the string being 1 inch longer, what is the strain caused by a weight of 2 lb. 6 oz.?

Exercise 10.

A string 55 inches long is attached to two hooks 25 inches apart on one level, and a weight of 6000 grains is hung on the string at 30 inches from one end; required the strains on the two parts of the string.

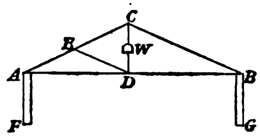
Exercise 11.

The distance of the hooks is 17·1, the length of the string 18·9, and a weight of 240 oz. is hung on at 8·5 from one end of the string; required the tension.

Exercise 12.

A rope 117·9 long is tied to two hooks 117·3 apart, and a weight of one hundredweight (112 lb.) is hung on at the distance 39 inches from one end; required the strains.

The same method may be applied to the case of a simple roof consisting of two rafters A C, B C set on the tops of the walls and meeting at C. But in this case we have to consider how the thrusts of the two rafters are to be resisted at A and B. It is quite clear that the tendency is to push the walls outwards, and thus, unless the walls be very massive or the roof very light, the building may be thrown down. In order to resist this outward thrust, we may employ a tie-rod from A to B, and then the walls have only to support the downward pressures.



At the point A there are three pressures, one the oblique thrust C A which we can compute, and which we may hold as known; the second the *pressure* caused by the tie-rod A B against the lower end of A C; the third being the upward elasticity of the wall; and these must clearly be proportional to the sides of the trigon C A D, so that the tension on the tie-rod, and the share of the load borne by the wall A F may be easily computed.

In the same way we may proceed at the point B; and, if our computations be correct, we shall find the same tension B A as before, while the two vertical pressures at A and B make up the whole weight W.

Exercise 13.

The span of a roof being 42 feet, the rise in the middle 20 feet, what are the strains caused by one ton hung from the ridge?

Exercise 14.

If the span were 80 feet and the rise at the centre only 9 feet, what strains would be caused by one ton hung from the ridge?

Exercise 15.

The span being 28 feet, and the rise at 11 feet from the front wall being 7 feet 1 inch; required the strains caused by a weight of one ton on the ridge.

Exercise 16.

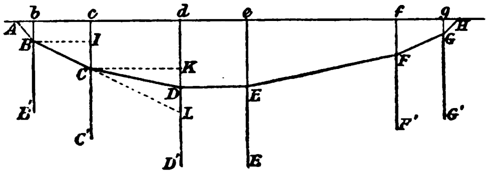
The whole span being 32 feet 7 inches, and the rise 10 feet at 15 feet 2 inches from one wall; required the strain for each ton suspended from the ridge.

EXERCISE 17.

If the span were 134 feet 3 inches and the rise 10 feet at 59 feet 7 inches from one wall, what would be the strains due to one ton hung from the ridge?

The student will observe that in all these calculations we have omitted the weights of the parts, such as the rafters and cords. This omission must be supplied hereafter.

When several weights are suspended along a string we may trace the strains from one point to another by considering that the tension on any part such as C D, represents a pressure at C in the direction C D and a like pressure at D in the opposite direction D C; and when the form assumed by the string is known, the proportions of all



the strains may be computed, so that if one of them be given the rest may be found. As an example we shall suppose that A B C D E F G H represents a cord attached to hooks at A and H on the same level, and having weights hung on at the points B, C, D, E, F and G; the dimensions of the figure being Ab=4, Ac=19, Ad=43, Ae=60, Af=100, Ag=112, AH=115, while bB=3, cC=11, dD=18, eE=18, fF=9, and gG=4; we shall also suppose that a weight of 29 ounces is hung on at C.

For the purpose of computing the other strains at the point C we must form a trigon having its sides parallel to

C D, C C′ and C B. This may be done by continuing B C to meet D D′ in L, or by producing D C to meet B B′. The lengths of the sides of C D L may be computed from the data, because when B I and C K are drawn parallel to A H, C K L is similar to B I C; and when these lengths are known the tensions of C D and C B are easily found. By following a similar course at the point B we can compute the weight hung on at B, and the tension of the string A B. Similarly we may proceed to D, thence to E and so on.

The student may make these computations; the dimensions have been given so as to avoid fractions in all the results. When we shall have acquired some knowledge of trigonometry or the calculation of angles, we shall be able to make this investigation much more neatly. After having computed the weight it may be worth his while to set up the actual arrangement.

Exercise 18.

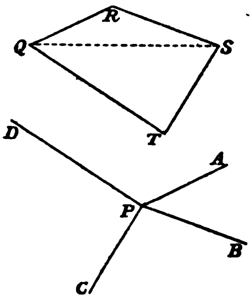
The ends of a cord are fastened to the two hooks A and L, required the weights which must be hung on at its several points in order that it may take the form given by the dimensions $A b = 22$, $b B = 120$; $A c = 66$, $c C = 237$; $A d = 132$, $d D = 349$; $A e = 220$, $e E = 454$; $A f = 330$, $f F = 550$; $A g = 465$, $g G = 634$; $A h = 585$, $h H = 684$; $A i = 705$, $i I = 719$; $A k = 885$, $k K = 738$; $A l = 1000$, $l L = 738$; when a weight of 418 grains is hung from K. Required also the strains on the several parts of the cord. Verify also the results by actual construction; counting in tenths of inches, the outline may be marked on a wall, and nails may be driven in at the points A and L so that the weights may hang free.

LESSON XLIV.

ON THE BALANCE OF FOUR PRESSURES.

When three pressures balance each other at a point, the angles made by them are determinate; the balance can take place in only one way. But when more than three pressures resist each other, the angles are not determined by their intensities, but may be varied within certain limits; and conversely, when the angles are given the ratio of the pressures may be varied.

Let us suppose that the point P is kept at rest by four pressures represented by P A, P B, P C, P D. In the neighbourhood draw Q R parallel and equal to P A, R S parallel and equal to P B, and join S Q; this line S Q represents a pressure which when applied at P would resist the two pressures P A and P B, wherefore its opposite Q S may be taken as equivalent in effect to these two. Draw now S T parallel and equal to P C, then, in the same manner, a pressure represented by T Q would resist the two pressures Q S and S T, or would resist the three pressures P A, P B, P C, and therefore P D drawn parallel and equal to T Q must represent the fourth pressure acting at P.



Hence we see that when four pressures mutually resist each other at a point, they are parallel and proportional to the four sides of a tetragon; and it is clear that the theorem may be extended to any number of pressures, or that *pressures* parallel and proportional to the side of

polygon (taken in order continuously) applied at a point, mutually resist each other. The theorem concerning three pressures is thus only one case of a much more comprehensive theorem.

Exercise 1.

Three pressures P A, P B, P C of 4, 5 and 6 ounces respectively acting on a point P at the angles $APB=100°$, $BPC = 110°$ are resisted by a fourth pressure; required its direction and intensity.

Exercise 2.

Four pressures $PA = 23$, $PB = 20$, $PC = 34$, $PD = 38$, all in one plane, make angles $APB = 40°$, $BPC = 63°$, $CPD = 95°$ act on a point P; required the directions and intensity of the pressure which resists them.

A figure of three sides is necessarily in one plane; three pressures can only resist each other when their directions are in one plane. But the four lines Q R, R S, S T, T Q may not be in one plane and yet the balance may hold good, so that we must still further extend the theorem and say that if any closed series of lines be formed whether all in one plane or not, pressures acting on a point, and being parallel and proportional to these lines taken in order, mutually resist each other.

Hence if three pressures parallel and proportional to the three sides A B, A D, A E (figure, Lesson XIV.) of a rhomb act upon a point, they are resisted by a pressure parallel and proportional to the diagonal G A; or we may say that their joint action is equivalent to that of a pressure parallel and proportional to A G.

EXERCISE 3.

Three pressures of 52, 53, 56 act in directions perpendicular to each other; required the intensity of the resisting pressure.

EXERCISE 4.

A pressure of 70 lb. is directed northwards, one of 50 lb. eastward and a third of 49 lb. straight upwards, what single pressure would resist them?

EXERCISE 5.

What single pressure resists three pressures 36 lb., 36 lb., and 73 lb. directed parallelly to the sides of a cube?

EXERCISE 6.

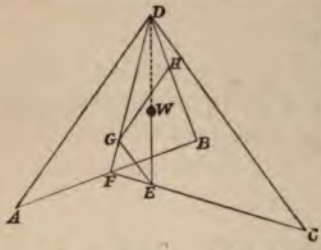
Required the equivalent to three pressures of 33, 38 and 66 lb. acting in directions at right angles to each other.

LESSON XLV.

ON THE TRIPOS.

IN the case of the simple roof treated of in Lesson XLIII., the slightest inclination sidewise would cause a fall. For the purpose of suspending a weight W safely we must have three legs resting on three points not in one straight line, and meeting in one point. Let A, B, C be three such points on the ground, and AD, BD, CD, the three legs meeting at the apex D; and, a weight W being suspended from D, let us endeavour to investigate the strains thereb *occasioned on the* three legs.

Producing the line D W to meet the surface of the ground in E, we may assume the length of this normal to represent the weight; now this pressure D E downwards is resisted by three pressures in the directions A D, B D, C D; selecting two of these as A D, B D, we observe that the single pressure equivalent to them must be in the plane A D B; while the single pressure equivalent to D E and C D must be in the plane C D E;



now these two equivalent pressures must be in the same straight line but in opposite directions, wherefore it must be in the intersection of these two planes. Hence if C E be continued to meet A B in F, the equivalent pressures must be in the straight line D F. Draw then E G parallel to C D, and it must represent the thrust on the leg D C, while G D represents the single equivalent to the thrusts on D A and D B. Draw G H parallel to A D, then H G represents the thrust on D A, D H the thrust on D B.

When the distances A B, B C, C A, and the lengths of the legs C D, A D, B D are given, we can compute the normal D E and the position of E, from which again the position of F and the line D F may be found. The calculation is somewhat laborious when the structure is scalene; but if A B C be an equilateral trigon, as is usually the case, and if the three legs be of one length the computation is short and easy. The student should work the whole out for himself; and, after the examples he has had of such investigations, he should be able to solve the following *exercises without* further help.

Exercise 1.

When the six distances A B, B C, C A, A D, B D, C D are all alike, what strains are caused by a weight of 198 lb. hung from D ?

Exercise 2.

The distances A B, B C, C A being each 30 feet, while A D, B D, C D are each 40 feet, required the strains caused by a weight of 1440 lb. hung from D.

Exercise 3.

When $AB = 39$, $BC = 41$, $CA = 43$; $AD = 50$, $BD = 52$, $CD = 53$, required the strains caused by 1000 lb. suspended from D.

APPENDIX.

SPECIFIC GRAVITIES.

Acetic Acid	1·063
Agate	2·590
Alabaster, Ordinary	2·730
Alcohol, Absolute	0·794
„ Proof	0·919
Aloes	1·380
Aluminum	2·550
Amber	1·030
Antimony	6·719
Arsenic	5·763
Asbestos	2·913
Ashwood	0·800
Asphaltum	1·069
Barium	4·000
„ Chloride	3·820
„ Nitrate	3·284
„ Sulphate	4·230
Basalt	2·864
Beechwood	0·852
Beeswax	0·935
Benzole	0·899
Bismuth	9·882
Bone, Ox	1·654
Boracic Acid	1·479
Borax	1·714
Boron	2·680
Boxwood	1·030
Brass, Fine	8·350
„ Cast	8·050
Brazilwood	1·031
Brick	2·000
Cadmium	8·641
Calcium	1·584
Camphor	0·989
Cane Sugar	1·854
Caoutchouc	0·933
Carbon	1·8[illegible]5
Carbonic Acid	0·002
Chalk	2·794
Cherry Tree	0·715
Chloroform	1·495
Chromium	6·810
Cinnabar	8·098
Clay	2·160
Coal	1·370
Cobalt	8·710
Cocoawood	1·040
Copper, Cast	8·788
„ Wire	8·878
Cork	0·240
Diamond	3·520
Ebony	1·177
Elm	0·600
Ether, Sulphuric	0·739
Fat, Hog	0·937
Firwood	0·550
Flint	2·594
Fluor Spar	3·170
Galena	7·585
Glass, Bottle	2·733
„ Flint	3·310
Glycerine	1·280
Gold, Fine	19·640
„ Standard	18·888
Granite	2·737
Grape Sugar	1·606
Gun Metal	8·784
Gunpowder	1·745
Heavy Spar	4·230
Honey	1·450
Horn	1·840
Hydrochloric Acid	1·2[illegible]

SPECIFIC GRAVITIES—*continued.*

Ice	0·918	Platinum	19·500
Iceland Spar	2·700	„ Rolled ..	22·069
Indigo Blue	1·350	Plumbago	2·255
Iodine	4·948	Plumwood	0·785
Iridium	21·815	Poplar	0·383
Iron, Bar	7·788	Porcelain	2·385
„ Cast	7·207	Portland Stone	2·496
„ Pure	7·880	Potassium	0·865
Ivory	1·825	Pumice Stone	0·914
Jasper	2·359	Pyrites, Copper	4·954
Juniper Wood	0·556	„ Iron	3·440
Lancewood	0·900	Quartz	2·655
Lead	11·445	Resin	1·100
„ White	3·160	Rotstone	1·981
Lignum Vitæ	1·327	Silicon	2·493
Lime	2·750	Silver, Fine	11·090
Lithium	0·583	„ Standard ..	10·535
Magnesium	1·743	„ Chloride	5·501
Mahogany	1·063	„ Iodide	5·655
Manganese	7·172	„ Nitrate	4·347
„ Oxide ..	3·647	Slate, Common	2·672
Maplewood	0·755	Soda Alum	1·604
Marble, White	2·717	Sodium	0·972
Mercury	13·568	Spelter	7·065
Mica	2·862	Starch	1·505
Milk, Cow's	1·030	Steel	7·850
Molybdenum	8·620	Strontium	2·544
Mulberry Wood	0·897	Sulphur	2·087
Naphtha	0·700	„ Fused	1·991
Nickel	7·807	Sulphuric Acid	1·840
Nitre	2·070	Tartaric Acid	1·700
Nitric Acid	1·420	Tin, Pure	7·471
Nitroglycerine	1·595	„ Ordinary	7·320
Oak	0·925	Titanium	5·300
Oil, Almond	0·915	Turpentine	0·890
„ Cinnamon	1·044	Uranium	18·375
„ Clove	1·036	Verdigris	3·572
„ Cod	0·923	Vermilion	4·230
„ Lavender	0·894	Walnut Wood	1·617
„ Linseed	0·936	Water, Distilled ..	1·000
„ Olive	0·908	„ Sea	1·030
„ Whale	0·923	Willow Wood	0·585
Pearwood	0·661	Yew Wood	0·788
Peat	0·600	Zinc, Hammered ..	7·191
„ Hard	1·329	„ Pure	6·861
Pewter	7·471	„ Sulphate	2·297
Phosphorus	1·763	Zirconium	4·150
Pitch	1·150		

SPECIFIC GRAVITIES—*continued.*

Alloys (one Atomic Weight of each).		Bismuth and Tin ..	8·772
		„ Zinc ..	9·046
		Copper and Lead ..	10·375
Aluminum and Copper	5·731	„ Silver ..	9·904
„ Silver	8·744	„ Tin	8·056
„ Tin ..	5·454	„ Zinc ..	8·019
„ Zinc ..	4·532	Gold and Lead	14·466
Antimony and Bismuth	8·378	„ Silver	14·870
„ Copper	7·990	„ Tin	11·833
„ Lead ..	8·989	Lead and Platinum ..	15·770
„ Zinc ..	6·929	„ Silver	11·054
Bismuth and Copper	9·634	„ Tin	9·426
„ Gold ..	13·403	„ Zinc	8·828
„ Lead ..	10·943	Tin and Zinc	7·188
„ Silver ..	10·068	Iridium and Osmium	19·386

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